

Quantum Theory in the Rigged Hilbert Space—Irreversibility from Causality ¹

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Abstract

After a review of the arrows of time, we describe the possibilities of a time-asymmetry in quantum theory. Whereas Hilbert space quantum mechanics is time-symmetric, the rigged Hilbert space formulation, which arose from Dirac's bra-ket formalism, allows the choice of asymmetric boundary conditions analogous to the retarded solutions of the Maxwell equations for the radiation arrow of time. This led to irreversibility on the microphysical level as exemplified by decaying states or resonances. Resonances are mathematically represented by Gamow kets, functionals over a space of very well-behaved (Hardy class) vectors, which have been chosen by a boundary condition (outgoing for decaying states). Gamow states have all the properties that one heuristically needs for quasistable states. For them a Golden Rule can be derived from the fundamental probabilities $\mathcal{P}(t) = \text{Tr}(\Lambda(t)W^{\text{Gamow}}(t_0))$ that fulfills the time-asymmetry condition $t \geq t_0$ which could not be realized in the Hilbert space.

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1 Preface—Time-Asymmetries

This preface was added to the paper in order to explain what we mean by the word irreversible in the title.

The term “irreversible” has two different uses and has been applied to several different phenomena [1, 2]. These different phenomena are also called different “arrows of time”. These arrows are not unrelated to each other, but there is no consensus as to their exact relationships. Somehow, all these arrows seem to be connected to the vague, intuitive notion of causality.

Most of the time the word “irreversibility” is used to refer to these arrows of time, i.e. to the directedness of the time evolution of a physical system or of the state of a physical system, classical or quantum. An alternate use of the word “irreversible” is to describe non-invariance (of the observables) with respect to the (antilinear) time reversal transformation A_T . These two notions, though related are not the same and must be distinguished. Whereas the time reversal operator is defined by its relations to the other *observables* that generate the space-time symmetry transformations or by its relations with these transformations (Poincaré group, Galilean group), “irreversibility” above means the impossibility or improbability to create a *state* which evolves backward in time. Irreversible time evolution is not necessarily in contradiction with a T -invariant Hamiltonian if a mathematical theory is used that makes a distinction between states and observables. Time reversal invariance—or its violation—is not the key for understanding irreversibility. We shall mention A_T transformations only briefly in Sect. 7.1. In this paper and in the title, irreversibility is the time-asymmetry due to a preferred direction of time evolution.

Of the different irreversible phenomena or the different “arrows of time”, most prominent and oldest is the thermodynamic arrow of time (TA): the entropy \mathcal{S} in an isolated (classical) physical system increases, $\frac{d\mathcal{S}}{dt} > 0$, until it reaches equilibrium, $\frac{d\mathcal{S}}{dt} = 0$ (2nd law of thermodynamics). Using (for the sake of simplicity) the time-symmetric equation of Newtonian mechanics for the scattering of molecules in a gas, it has been shown [3] that Boltzmann’s Stosszahl-Ansatz implies assumptions (boundary conditions) from which the TA follows: “If one assumes that the gas was prepared in some manner in the past..., then it follows that correlations between molecules and scattering centers will arise only from past but not from future collisions,” (R. Peierls in [3]). That initial conditions, not final conditions, are specified is part of the intuitive law of causality.

Another arrow of time of classical physics is the radiation arrow (RA). The Maxwell equations, like all local physical laws, are symmetrical in time, yet retarded radiation predominates. The phenomenological law according to which nature favors the retarded potential over the advanced solution of the Maxwell field equations is called the time arrow of radiation. An effective way to describe this time arrow is to formulate an additional axiom to

the Maxwell equations—a particular, time-asymmetric, boundary condition. This boundary condition excludes the strictly incoming fields A_{in}^μ (Sommerfeld radiation condition). For a system of charged particle, the fields $A^\mu(x) = A_{adv}^\mu + A_{out}^\mu = A_{ret}^\mu + A_{in}^\mu$ acting on a particle at x , in *every* region of space-time, are only the retarded fields of the other particles in the region, $A_{in}^\mu = 0$. In other words, all fields possess advanced sources somewhere in the universe; *Radiation must first be emitted by a source before it can be detected by the receiver*. Other boundary conditions are imaginable, e.g. the time-symmetric boundary conditions for which all fields also possess sinks and will be absorbed somewhere. In the famous Einstein-Ritz arguments [4], Einstein thought that physics could be restricted to the time-symmetric case for which retarded and advanced fields are equivalent. Then the RA is based on probability (i.e., the TA). Ritz considered this restriction of the boundary conditions as not allowed, in which case phenomena demand the choice of the retarded fields as the only possibility and the TA has its origin in the RA.

The classic mathematical attempt to settle this argument and account for the predominance of retarded radiation was given by Wheeler and Feynman[5]. They start with time-symmetric boundary conditions for the field equations (time-symmetric electrodynamics) and introduce a cosmic absorber (of huge amounts of dust matter) which annihilates the advanced fields over the retarded. This gives the impression that “ the physics of radiation can be regarded as, at bottom, time-symmetric with only the statistics of large numbers giving the appearance of asymmetry ” (J.A. Wheeler in [1]). However such an absorber has to contain some arrow of time. In order to provide the appropriate thermodynamic conditions for absorption to occur in the far future, a random initial state for the matter has to be postulated, and this initial state depends in turn on the retardation. Therewith one is back to the Einstein-Ritz controversy [4] as to whether irreversibility is exclusively based on probability considerations, $TA \Rightarrow RA$, (which Einstein believed) or whether an initial condition and thus causality is the basis of irreversibility, in which case $RA \Rightarrow TA$ (Ritz’s opinion). Up to today, there seems to be no agreement in the literature as to whether $TA \Rightarrow RA$ or $RA \Rightarrow TA$, though the former is more prevalent [1]. Causality and probability may just be two aspects of one and the same principle.

For every situation with increasing entropy and retarded potentials one also has a completely time reversed situation with decreasing entropy and advanced potentials. This dichotomy is not only restricted to the electromagnetic field but can also be found in quantum physics. This is not surprising because the quantum theory of radiation cannot be expected to be free of an arrow of time if classical electrodynamics possesses one. However, the mathematical theory of the Hilbert space allows only time-symmetric boundary conditions. To accommodate time-asymmetric boundary conditions one has to extend the mathematical theory beyond the confines of the Hilbert space.

In addition to the two arrows of classical physics of relevance to time scales achievable

in the laboratory, a third arrow exists at the cosmic time scale—the cosmological arrow of time (CA). This states that the universe expands (the contracting solutions of the equation of motion are excluded) or at least that we live in the expanding phase fairly close to the initial singularity of the big bang.

Usually the cosmological arrow is believed to be the master arrow from which the others follow [P.C.W. Davis in [1]]. Though there seems to be no consensus whether $CA \Rightarrow RA \Rightarrow TA$ or whether $CA \Rightarrow TA \Rightarrow RA$, the most attractive scenario is the entropy gap: the expansion of the universe during the first three minutes was much faster than the relaxation time of the nucleosynthesis, leaving the majority of the cosmological material for aeons in a metastable state. The quasistable states of our sun are the metastable stuff that is driving the time-asymmetric processes. On the other hand, local arrows of time need not depend on asymmetric cosmologies. For example, in some simple models [6], the light cones in a closed Friedmann universe tend to imitate the expansion figure of space-time. As a consequence, outgoing electromagnetic waves occur near the big bang and incoming (in our local sense of time) waves occur close to the big crunch.

The CA and the big bang give us a means of defining the cosmic time and its origin $t = t_0 \equiv 0$. In order to define irreversible time evolution (time evolution that does not extend to $-\infty$ and to $+\infty$) one has to be able to choose such a reference time $t_0 \neq \pm\infty$ like the creation time of the universe. In the laboratory one fixes this time $t_0 = 0$ for each physical system individually as, for example, the time at minimum entropy or the time at which the radiation is emitted. In quantum physics (the concern of this article) this time t_0 is the time at which the state has been prepared (e.g. a resonance has been “created”) and at which one can start the detection of an observable (e.g. of the decay products) in this state.

Before we turn to the quantum mechanical arrow of time we want to mention the psychological arrow. It is the arrow of time by which we remember the past and predict the future. It is physically not well-defined, but usually considered subsumed under the thermodynamic arrow which in turn is maintained by the expanding universe. Thus if in a (re)contracting universe the entropy decreases (which may or may not be the case) then the psychological arrow should also turn around to be again in the direction of entropy increase (Hawking in [1]). That is, in such a contracting universe—or contracting region of the universe—the direction of the psychological arrow should be the opposite of *our* subjective sense of time; initial conditions should turn into final conditions and vice versa. This then would be the realm of states with negative time (with $t = 0$ meaning the creation time of our universe), or the time beyond the “switch-over” point of maximal expansion, if we think of the cosmic time as cyclic. How this reversal of time could take place is hard to comprehend. It does not seem to be of much practical importance; probably it is just a feature of the time-symmetric differential equations, which are incomplete without boundary conditions. For the mathe-

mathematical boundary condition we have a choice, but for the physical boundary condition in our world we have not. The physics of our universe is time-asymmetric. It is important to have a mathematical theory that is not too restrictive and allows one arrow of time. Then the opposite arrow of time is obtained by time reversal transformation of the boundary condition. Whether for every physical system or process there is also a time reversed one in *our* world, is a different question which is not answered by the T- or CPT-invariance of the Hamiltonian (differential operator). Solutions of time-symmetric dynamical equations with time-asymmetric boundary conditions come in pairs. With the choice of the boundary condition, one of the two time-asymmetric solutions is selected. This applies to the classical equation of general relativity (big bang–big crunch; black hole–white hole) and electromagnetism (retarded–advanced) and must as well apply to the mathematical theory of quantum physics. One should not restrict the mathematics of quantum mechanics to such an extent that time-asymmetric boundary conditions cannot be formulated with mathematical rigor. The Hilbert space theory of quantum mechanics does not allow such a time-asymmetric formulation, while the rigged Hilbert space theory allows time-asymmetry in either of the directions (which are related by T (or CPT) conjugation, cf. Sect. 7.1 and [48]). The physics in our world chooses one of these directions.

In analogy to the radiation arrow of time, the quantum mechanical arrow of time can be formulated without reference to the mathematical theory as: *a state first must be prepared by a preparation apparatus before an observable can be detected in it by the registration apparatus*. We will call this the preparation→registration arrow of time [7]. The preparation→registration arrow of time is perhaps the closest we can come to a physical definition of the psychological arrow of time, because it is the experimentalist who first prepares the state and then activates the detector. Irreversibility that we mean in the title is the asymmetry of the time evolution based on this arrow of time. Its most prominent realization in quantum physics is the intrinsic time evolution of resonances and decaying states.

In contrast, conventional irreversibility in quantum physics has been considered to be extrinsic. Two forms of state changes are usually discussed in quantum mechanics:

1. A unitary time evolution generated by the Hamiltonian H of the quantum system S :

$$\rho \rightarrow \rho(t) = e^{-iHt} \rho(0) e^{iHt} = U^\dagger(t) \rho U(t), \quad -\infty < t < \infty \quad (1a)$$

or

$$\phi(t) = e^{-iHt} \phi, \quad -\infty < t < \infty \quad (1b)$$

where ρ is the statistical or density operator of a mixed state and ϕ the state vector if the state is pure, $\rho = |\phi\rangle\langle\phi|$.

This change of the state is reversible and could be equivalently described by the Schrödinger equation

$$i \frac{d\phi}{dt} = H\phi \quad (1c)$$

with the “Hilbert space” boundary condition $\phi(t) \in \mathcal{H}$.

2. The “reduction of the state vector” on measurement, which in von Neumann’s idealization is given by the following additional axiom (“collapse of the wave function”—ideal measurement of 1st kind): If P_{a_i} are the projection operators of the observable measured $A = \sum_i a_i P_{a_i}$, then

$$\rho \rightarrow \rho^{\text{after}}(t_0) = \sum_i P_{a_i} \rho(t_0) P_{a_i} \quad (2a)$$

or, if the results of the measurement is a_j ,

$$\rho \rightarrow \rho^{\text{after}}(t_0) = P_{a_j} \rho(t_0) P_{a_j}. \quad (2b)$$

Here ρ^{after} is the state after the measurement. If one defines the (von Neumann) entropy by

$$\mathcal{S}(\rho) = -\text{Tr}(\rho \ln \rho) \quad (3)$$

then for (2a)—but not for (2b)—one obtains: $\mathcal{S}[\rho^{\text{after}}(t_0)] > \mathcal{S}[\rho^{\text{before}}(t_0)]$, i.e. this entropy increases if the initial entropy is fixed and smaller than the maximal entropy. If the initial state in (2a) was a pure state $\rho = |\phi\rangle\langle\phi|$, then the state after measurement is a mixture:

$$|\phi\rangle\langle\phi| = \sum_i \sum_j |a_i\rangle\langle a_i| \langle\phi|a_j\rangle\langle\phi|a_i\rangle \rightarrow \rho^{\text{after}} = \sum_i |a_i\rangle\langle a_i| \langle\phi|a_i\rangle\langle\phi|a_i\rangle. \quad (4)$$

This decoherence (elimination of the interference terms $\langle a_i|\phi\rangle\langle\phi|a_j\rangle$, $i \neq j$) is considered an irreversible process, though no time has actually elapsed.

The increase in von Neumann entropy is conventionally considered as the arrow of time in quantum mechanics. It is not due to time-asymmetric laws (equations and/or boundary conditions) but appears through the extraneous act of measurement. The intrinsic time evolution of a quantum physical system, i.e. the time evolution of a state of a “closed physical system”, isolated from all external influences, is described in the Hilbert space by (1) and is reversible.

The change described by (2) leads to the following problem: if $\rho = \rho(t)$ is time dependent and given by (1a) (Schrödinger picture) then A and P_{a_i} are time independent and ρ^{after} in (2) is at the same time as $\rho(t)$. The collapse (2) is supposed to happen instantly and would not shed any light on the question of a time arrow. However, realistically, every measurement takes time. To avoid this inconsistency one considers the Heisenberg picture [8]: keep ρ time independent $\rho = \rho(t_0 = 0)$ and equal to the “initial” state of (1), and take for the projection operators the time evolved $P_{a_i}(t) = e^{iHt}P_{a_i}e^{-iHt}$. Then in place of (2b) one has

$$\rho^{\text{eff}}(t_a) = P_a(t_a)\rho P_a(t_a), \text{ with the condition that } t_0 < t_a \quad (5)$$

(or at least $t_a \geq t_0$, but $t_a = t_0$ would be the unrealistic case of instant measurement). The change (5) combines the two conventional changes of states in quantum mechanics. Its generalization for a sequence of observables, $A(t_a)$, $B(t_b) = \sum_j b_j P_{b_j}(t_b)$, etc. to a “history” [8] $[\mathbf{P}] = (P_a(t_a)P_b(t_b) \cdots P_c(t_c))$ is:

$$\rho^{\text{eff}}(t_c) = \sum_{i,j,\dots,k} P_{c_k}(t_c) \cdots P_{b_j}(t_b) P_{a_i}(t_a) \rho(t_0) P_{a_i}(t_a) P_{b_j}(t_b) \cdots P_{c_k}(t_c). \quad (6a)$$

with the condition

$$t_0 < t_a < t_b < \cdots < t_c \quad (6b)$$

Since the times “after” t_a, t_b, \dots, t_c cannot precede the “initial” times $t_0 < t_a, t_a < t_b \dots$ one obtains the time ordering (6b). In contrast to (2b), where the change is to happen instantly, for (5) and (6a) one can talk of a change in time. This time ordering follows from the idea that there is a knowable initial (prepared) state ρ from which the series of probabilities

$$\mathcal{P}(a, t_a) = \text{Tr}(P_a(t_a)\rho) = \text{Tr}(P_a(t_a)\rho P_a(t_a)) = \text{Tr}(\rho^{\text{eff}}(t_a)), \quad t_a > t_0 \quad (7a)$$

$$\mathcal{P}(b, t_b; a, t_a)$$

$$= \text{Tr}(P_b(t_b)P_a(t_a)\rho P_a(t_a)P_b(t_b)) = \text{Tr}(P_b(t_b)\rho^{\text{eff}}(t_a)), \quad t_b > t_a > t_0, \quad (7b)$$

and in general

$$\mathcal{P}(c, t_c; \cdots; b, t_b; a, t_a) = \text{Tr}(P_c(t_c) \cdots P_b(t_b)P_a(t_a)\rho P_a(t_a)P_b(t_b) \cdots P_c(t_c)) \quad (7c)$$

can be *predicted* (rather than *retrodicted*). Though it may be an oversimplification to think of $\rho^{\text{eff}}(t)$ as the state of the system after the measurement of the observable P_a , the $\text{Tr}(\rho^{\text{eff}}(t_a))$ are meaningful physical quantities, namely the probabilities to observe with an apparatus the value a for the observable A (cf. Sect. 7.4 below).

The time ordering (6b) expresses an arrow of time. If one considers an experiment performed on a quantum system in the laboratory with the state ρ prepared by the preparation apparatus (accelerator) and $P_a(t)$ the observable registered by the registration apparatus (detector), then the condition $t_0 < t_a$ in (7a) expresses the preparation→registration arrow of time that we mention above and discuss in greater detail in Sect. 6 below, cf. eq. (40). This intuitively obvious time ordering and the arrow of time it implies is the basis of the quantum mechanical irreversibility discussed in our article. The initial time $t_0 = 0$ is the time at which the preparation of the state ρ is completed and the registration of the observable $P_a(t)$ can begin. The probability $\mathcal{P}(a, t_a)$ is the probability to register the observable $P_a(t_a)$ at the time t_a in a prepared state ρ . For example, if ρ describes the prepared state of a $K^0 - \bar{K}^0$ system and P_a the projection operator on the $\pi^+\pi^-$ subspace then $\mathcal{P}(a, t_a)$ is the probability for decay of the neutral K meson into $\pi^+\pi^-$, which can be detected only at a time $t_a > t_0$. Since the formation time scale of the neutral K is ten orders of magnitude less than that of the decay, the creation time t_0 is very precisely fixed.

The formula (7a) of the quantum mechanics of measured systems has been generalized in [8] to (7a), (7b), ..., (7c) which have been understood to apply to the quantum theory of cosmology. The probabilities are the probabilities of histories $P_a(t_a)$, $P_a(t_a)P_b(t_b)$, $\dots P_a(t_a)P_b(t_b)\dots P_c(t_c)$ of such systems as our universe as a whole. The arrow of time expressed by the time ordering in (6a) and (6b) would in this case be the time-asymmetry of the universe, i.e. the cosmological arrow of time.

Nothing has been said about the probabilities (7) for $t_a < t_0, \dots, t_c < \dots < t_b < t_a < t_0 = 0$. In cosmology, where t_0 is the time of the big bang, we would not know. In the decay process of a laboratory experiment we could make the following conclusion: the detector does not click if it is switched on before the decaying state has been prepared at $t = 0$. It is therefore reasonable to assume that the probabilities (7) are zero for these negative values of t .

As will be discussed in Sect. 4 below, in the Hilbert space theory (with no other further assumptions than H is self-adjoint and semi-bounded) the probabilities of (7) are identically zero (for all t) if they are zero on a negative time interval (actually on any set of non-zero measure). Therefore one cannot implement the above program as a mathematical theory in the Hilbert space unless the probabilities $\mathcal{P}(a, t_a); \mathcal{P}(b, t_b; a, t_a) \dots$ are defined to be different from zero for all $t_a < t_0, t_b < t_a$, etc. The Hilbert space does not allow for a theory in which both energy *as well as* time have a lower bound (creation time). If one wants to have a quantum mechanical arrow of time, i.e. a mathematical formulation of the obviously correct preparation→registration arrow, then one has to go beyond the Hilbert space formulation. This is the subject of the present paper indicated by the word rigged Hilbert space (also called Gelfand triplet) in the title.

Irreversibility in conventional quantum theory is always thought of as being due to ex-

ternal influences upon the non-isolated (“open”) quantum system. The irreversible time evolution of open quantum systems is described by the master equation [9]

$$\frac{\partial \rho(t)}{\partial t} = L\rho(t) \quad (8)$$

where $\rho(t)$ represents the state of the open system S , and the Liouville operator L is given by

$$L\rho(t) = -\frac{i}{\hbar}[H, \rho(t)] + \delta H(\rho) \quad (9)$$

For $\delta H(\rho) = 0$, (8) with (9) is the von Neumann equation whose solution is the reversible time evolution of the isolated quantum system given by (1a). Equation (8) with (9) is the standard equation for extrinsic irreversibility under the influences of an external reservoir R (which could be, for example, a measuring apparatus) upon the system S . The term $\delta H(\rho)$ represents some complicated external effects of the reservoir R upon the quantum system S . Under particular assumptions about the term $\delta H(\rho)$ the irreversible time evolution of S can be shown to be described by a completely positive semigroup² generated by a Liouvillian L [10]):

$$\rho(t) = \Lambda(t)\rho(0), \quad \Lambda(t) = e^{Lt}, \quad \text{for } t \geq 0. \quad (10)$$

This is the conventional semigroup evolution of open quantum systems (Sudarshan-Gorini-Kossakowski-Lindblad semigroup) [11].

This time evolution described by a Liouvillian L , where $L\rho$ is not just the commutator with the Hamiltonian of the system, $\frac{1}{i\hbar}[H, \rho]$, has also been applied to the decaying K^0 -system. It can evolve a pure state $|\phi\rangle\langle\phi|$ into a mixture and has been called non-quantum mechanical [12].

The ‘Irreversibility’ that we mean in the title is not the non-quantum mechanical irreversibility described by (10) with (9) and it is not the increase of the von Neumann entropy (3) due to the collapse axiom (2) and (4)³. And again, the ‘Semigroup’ that we mean is

²According to A. Kossakowski (private communication) maps $\rho \rightarrow \rho(t) = \Lambda(t)\rho$ from the set density operators into itself could also describe “some kind of intrinsic irreversibility” if $\Lambda(t)$ is not completely positive, but only positive. But so far it is not clear whether these maps have any physical meaning.

³The measurement process which changes ρ to ρ' is a scattering process of a microsystem on a macrosystem (“measurement scattering” [7]) and does not actually fulfill the idealized measurement axiom (2). Every scattering experiment possesses an arrow of time; preparation must always precede registration. The arrow of time for the change of state due to measurement is thus a consequence of the preparation→registration arrow, G. Ludwig, private communication. “I know of no ‘other’ time arrow than the preparation→registration arrow.”

not the semigroup (10) generated by the Liouvillian (9). Our semigroup is the semigroup generated by the Hamiltonian of the isolated quantum system:

$$e^{-iH^\times t}, \quad t \geq 0 (= t_0), \quad (11)$$

where H^\times is the extension of a self-adjoint (semi-bounded) Hilbert space operator \bar{H} . The semigroup (11) is obtained from the same time-symmetric dynamical equations (the Schrödinger equation (1c) or the von Neumann equation) from which one obtained the unitary group in (1a) and (1b) by integration. However, whereas the unitary group evolution, (1a) and (1b), is obtained if one requires that the set of ϕ in (1c) are elements of the Hilbert space \mathcal{H} , the semigroup evolution (11) is obtained from (1c) if one requires that the set of allowed solution ϕ in (1c) are elements of a space Φ_+^\times that extends the Hilbert space \mathcal{H} to a Gelfand triplet $\Phi_+^\times \supset \mathcal{H} \supset \Phi_+$. The arrow of time expressed by the time evolution semigroup (11) thus arises mathematically from time-symmetric dynamical equations solved with time-asymmetric boundary conditions. These time-asymmetric boundary conditions which follow from the preparation→registration arrow of time, cannot be mathematically formulated in the Hilbert space; in \mathcal{H} the equations (1c) always integrate to the one-parameter group (1b).

The semigroup time evolution (11) describes irreversible time evolution on the microphysical level (if one interprets the solutions of the Schrödinger equation in the extension Φ^\times of \mathcal{H} as microphysical states). Examples of isolated (closed, autonomous) microphysical systems with irreversible time evolution abound in the real world. Reversible time evolution is a feature of only a minority of closed microphysical systems, e.g. the ground states of molecules and nuclei and the few stable elementary particles. There is a much larger number of decaying states and resonances (excited states of molecules and nuclei, weakly decaying elementary particles, hadron resonances) which are not less isolated than ground states; their time evolution is irreversible (which is sometimes mentioned in textbooks [13]).

It has been argued that a decaying state or resonance is something complicated. The difficulty is that in Hilbert space \mathcal{H} one cannot find a simple initial condition for it. However, in its extension Φ_+^\times there is a simple (pure) initial state $\rho_i = |E_R - i\Gamma/2\rangle\langle E_R - i\Gamma/2|$ at the time t_0 , the creation time of the resonance (e.g. the time when the K^0 is leaving the proton target in $\pi^- p \rightarrow \Lambda K^0$ of the many experiments on neutral Kaon decay and CP-violation). The decaying state vector $\psi^G = |E_R - i\Gamma/2\rangle \in \Phi_+^\times$ and is a generalized eigenvector of the self-adjoint Hamiltonian H with complex eigenvalues $z_R = E_R - i\Gamma/2$ associated to the second sheet pole of the S -matrix [14]. (A resonance is actually described by a pair of poles $z_R = E_R \mp i\Gamma/2$, a pair of vectors ψ_\mp^G in a pair of spaces Φ_\pm^\times evolving by a pair of semigroups, cf. Sect. 7.1).

The semigroup time evolution of the Gamow vector, is derived as [14]:

$$e^{-iH^\times t}\psi^G = e^{-iE_R t}e^{-\frac{\Gamma}{2}t}\psi^G, \quad \text{for } t \geq 0 \text{ only.} \quad (12)$$

This is a mathematical consequence of the (mathematical formulation of the) preparation→registration arrow of time and vice versa (derived using the Paley-Wiener theorem). The time-asymmetry of resonances expressed by the semigroup (12) is not the irreversibility of quantum mechanical measurements; resonances evolve in time and decay without being “looked at”. But it is identical with the arrow of quantum cosmology if (5) and (6a) are applied to the initial state of the universe. The irreversibility that we mean is the asymmetric time evolution on the quantum level whose basis is the causality principle, which in turn can be inferred from a very special state of the universe at the beginning of time.

2 Introduction—Dirac Kets

“Physical causality can be traced directly to the existence of a simple initial condition of the universe. But how does that initial condition enter into the theory?”

—Murray Gell-Mann⁴

A mathematical theory of physics cannot be deduced from experiments alone; it will necessarily have to involve some idealizations. Mathematicians like to work with mathematical structures that are complete—algebraically complete and, in particular, topologically complete. Only with such structures can they prove existence theorems. Experiments can never be complete in this sense; physicists have only a finite amount (albeit very large sometimes) of experimental data with which to work. Mathematicians generalize to infinity through various means. The topology⁵ of a linear space defines the meaning of convergence of infinite sequences and the topological completion of a space is the appendage to the linear space of the limit elements of all infinite convergent (to be precise, Cauchy) sequences. A finite number of experiments, even if the number is arbitrarily large, cannot tell how the convergence of infinite sequences of states should be defined. Therefore the choice of topology cannot be “deduced” from experiments and only the overall success of a mathematical theory can show that one topology is preferable to another.

This paper explores different choices of topological completions for the spaces of states and observables. One is given by von Neumann’s Hilbert space completion. The others—

⁴p. 216 of M.Gell-Mann, *The Quark and the Jaguar*, (W.H. Freeman, New York, 1994).

⁵ Webster’s dictionary gives three definitions of the word topology: (1) the study of those properties of geometric forms that remain invariant under certain transformations, as bending, stretching, etc.; (2a) the study of limits in sets considered as collections of points; (2b) a collection of open sets making a given topological space. Physicists usually associate definition 1. with the word topology, we here use only definition (2). Mathematical structures are combinations of three basic kinds of structures: algebraic, topological and ordering [16]. The rigged Hilbert Space is the completion of the same linear (algebraic) space equipped with three different topologies.

actually a multitude of topologies, one for each particular kind of quantum physical system—are conjectured from Dirac’s formalism.

Dirac [15] introduced such elements as bras $\langle E|$ and $\langle x|$, kets $|E\rangle$ and $|x\rangle$ and an algebra of observables generated by such fundamental operators as the Hamiltonian H , momentum P and position Q . Of these kets he demanded that they were eigenvectors

$$\begin{aligned} H|E\rangle &= E|E\rangle \\ Q|x\rangle &= x|x\rangle \end{aligned} \tag{13}$$

and that they formed a complete basis system, i.e. that every vector ϕ could be written as

$$\phi = \sum_{n=1}^{\infty} |E_n\rangle (E_n|\phi) + \int dE |E\rangle \langle E|\phi\rangle, \tag{14a}$$

$$\phi = \int dx |x\rangle \langle x|\phi\rangle \tag{14b}$$

or

$$\langle x'|\phi\rangle = \int dx \langle x'|x\rangle \langle x|\phi\rangle, \tag{14c}$$

where $(E_n|\phi)$, $\langle E|\phi\rangle \equiv \phi(E)$ and $\langle x|\phi\rangle \equiv \phi(x)$ (the wave function) were thought of as scalar products, $\phi(x) = (|x\rangle, \phi)$.

These intuitive constructions were the results of Dirac’s unconfined vision, but not well-defined mathematical objects. A mathematical structure to envelope all Dirac’s calculative tools was not available at that time. The delta function $\langle x'|x\rangle = \delta(x' - x)$ inspired the development of distribution theory by Schwartz [17] and the $|x\rangle$, $|E\rangle$ inspired the creation of the rigged Hilbert space by Gelfand and his school [18][18a].

In the following discussion, we leave aside the proper eigenvectors $H|E_n\rangle = E_n|E_n\rangle$ with discrete eigenvalues E_n , which are conventionally negative $E_n = -|E_n|$, and the energy continuum starts at zero so that the integral in (14a) extends from 0 to ∞ (there can also be discrete eigenvalues in the continuous spectrum and then these $|E_n\rangle$ with $E_n > 0$ would also be included in the discrete sum of (14a)). In the S -matrix they correspond to poles on the negative real axis of the first sheet (bound state poles). Both the discrete and continuous values for the energy must be bounded from below (stability of matter) and by convention the lower bound for the energy continuum is chosen to be zero.

The discrete basis vectors describe stationary states: $W = \sum_n w_n |E_n\rangle (E_n|$. The linear space Ψ^{disc} spanned by these eigenvectors of the discrete spectrum could also be completed

into a Hilbert space $\mathcal{H}^{\text{disc}}$ (or a rigged Hilbert space) which is orthogonal to the Hilbert space \mathcal{H} of the continuous spectrum. In the following, we will only consider the linear space and their completions Ψ , Φ , \mathcal{H} which belong to the absolutely continuous spectrum. The quantum mechanics of stationary states is not affected by our considerations here. Thus \mathcal{H} in this paper denotes the Hilbert space of the absolutely continuous spectrum, often denoted as $\mathcal{H}^{\text{a.c.}}$ in the literature.

3 Hilbert Space (HS) and Rigged Hilbert Space (RHS) Formulation of Quantum Mechanics

“I would like to make a confession which may seem immoral: I do not believe in Hilbert space any more.”

—John von Neumann⁶

The first attempt at a rigorous mathematical theory for quantum physics was provided by Weyl [19] and von Neumann [20] using the mathematics that was available at that time, the Hilbert space (HS). The HS is the completion of a linear scalar product space (also called pre-Hilbert space) with respect to the topology⁷ given by the norm $\|\phi\| = \sqrt{(\phi, \phi)}$.

This norm topology is one of many possible choices and cannot be deduced from physical observations of quantum mechanical states and the observables represented by the operators. The HS properties are:

1. The wave functions $\psi(x) \equiv \langle x|\psi \rangle$ representing the probability $|\langle x|\psi \rangle|^2 \Delta x$ to detect the particle state ψ within the position interval Δx or the energy wave function $\phi(E) \equiv \langle E|\phi \rangle$ representing the the energy distribution in a particle beam are given in the HS by a *class* of Lebesgue square integrable functions $\{\psi(x)\}$ or $\{\phi(E)\}$ that differ on a set of Lebesgue measure zero.

$$\begin{aligned}\psi \in \mathcal{H} &\Leftrightarrow \{\psi(x)\} \in \mathcal{L}^2(\mathbb{R}) \\ \phi \in \mathcal{H} &\Leftrightarrow \{\phi(E)\} \in \mathcal{L}^2(\mathbb{R})\end{aligned}\tag{15}$$

In a scattering experiment, $|\psi(x)|^2$ represents the detector size, location and efficiency and the $|\phi(E)|^2$ represents the energy resolution of the accelerator. Unlike the classes of (15), the resolution of an experimental apparatus is always given by a single *smooth*

⁶In a letter to G. Birkhoff, quoted in G. Birkhoff, Proceedings of Symposia in Pure Mathematics, Vol. 2, ed. R.P. Dilworth, (American Mathematical Society, Providence, Rhode Island, 1961), p. 158. The letter is date Nov. 13, and Birkhoff believes the year was 1935.

⁷cf. footnote (5).

function, e.g. $\psi(x) \in \mathcal{S}$ (Schwartz space), and not by a class $\{\psi_1(x), \psi_2(x), \psi_3(x), \dots\}$ of functions which can vary wildly on any set of Lebesgue measure zero, e.g. on the set of all rational numbers.

One can always choose the one smooth function $\psi(x) \in \mathcal{S}$ in this class $\{\psi(x)\} \in \mathcal{L}^2(\mathbb{R})$, but the set of smooth functions (or more precisely the set of classes of Lebesgue square integrable functions containing a smooth function $\psi(x) \in \mathcal{S}$) is not HS-complete. For the HS-completeness, one needs the Lebesgue integral [21].

The notion of Lebesgue integrability is physically counterintuitive. Physicists make a finite number of measurements and interpolate smoothly between the experimental data. They never compute Lebesgue integrals, but calculate Riemann integrals using the smooth functions associated to their experimental apparatus.

2. The most fundamental commutation relations, such as $[P, Q] = -i1$ of momentum P and position Q , imply that these observables cannot be represented by continuous (and hence bounded) operators in HS. Therefore P and Q have only a limited domain of definition in \mathcal{H} . Physicists work with operators that one can add and multiply (Dirac's algebra of observables) and not with operators that cannot be defined on the whole space. Often such an algebra is the enveloping algebra of a symmetry group which represents physical transformations, for example, of the registration apparatus (detector) relative to the preparation apparatus (accelerator). One can always find a subspace $\Phi \subset \mathcal{H}$ on which these operators are defined (e.g. the space of differentiable vectors or the space of analytic vectors). Then one can define a topology⁸ with respect to which these operators are continuous. The space Φ cannot be a Hilbert space unless it is a finite-dimensional representation space of a compact group. However, such a subspace Φ can be represented a space of the smooth functions, in many cases (e.g. for Q or P) by the space \mathcal{S} :

$$\phi \in \Phi \Leftrightarrow \phi(x) \in \mathcal{S}. \quad (16)$$

3. The HS does not contain eigenkets like $|E\rangle$ and $|x\rangle$ or the bras $\langle E|$ and $\langle x|$, with the properties (13) and (14). Physicists use them however, e.g. as scattering states, and (14) is the fundamental relation for computations in quantum theory. The kets provide an opportunity to describe states of single microphysical systems.

In von Neumann's formulation with $\phi, \psi \in \mathcal{H}$, ϕ represents the state of an ensemble and $|\psi\rangle\langle\psi|$ represents the observable on an ensemble. The probability to measure the observable $|\psi\rangle\langle\psi|$ in the ensemble ϕ is $|(\psi, \phi)|^2$. The vectors ψ and ϕ are defined by

⁸cf. footnote (5).

the experimental apparatuses. For example, if ϕ is the idealized representation of a beam of microphysical particles prepared by an accelerator (idealized, because real accelerators prepare mixtures $\sum |\phi_i\rangle\langle\phi_i| = W$), then

$$\frac{\int_{\Delta E} dE |\phi(E)|^2}{\int_{\text{all } E} dE |\phi(E)|^2} \quad (17)$$

is the fraction of this large number of particles that have energy in the interval ΔE . One never talks of a single microphysical particle but always of a large number, an ensemble. (Equivalently one can talk of an ensemble of experiments.) The notion of a single microsystem or a state of a single microsystem does not exist in the standard HS formulation; ϕ describes an ensemble and $\phi(E) \in \{\phi(E)\} \in \mathcal{L}^2(\mathbb{R}_+)$ represents the energy distribution of that ensemble for which the physicists always choose a smooth function $\phi^{\text{smooth}}(E)$. Physicists do not work with the *class* of Lebesgue square integrable functions $\{\phi_1(E), \phi_2(E), \dots, \phi^{\text{smooth}}(E)\}$ which contains the smooth function. They cannot even isolate experimentally those classes of functions which do not contain a smooth function $\phi(E) = \phi^{\text{smooth}}(E) \in \mathcal{S}$, i.e. the elements of \mathcal{L}^2 which are not elements of \mathcal{S} . That means the class $\{\phi(E)\} \in \mathcal{L}^2(\mathbb{R}_+)$ which does not contain a $\phi^{\text{smooth}}(E)$ has no physical meaning.

The probability amplitude $\langle\psi|\phi\rangle$ of the physicist is therefore given by

$$\begin{aligned} \langle\psi|\phi\rangle &= \sum_{i=1}^{\infty} \langle\psi|i\rangle\langle i|\phi\rangle = \int_{\text{Riemann}} dE \langle\psi|E\rangle\langle E|\phi\rangle = \int_{\text{Riemann}} dx \langle\psi|x\rangle\langle x|\phi\rangle \\ &= \int \int_{\text{Riemann}} dx dE \langle\psi|x\rangle\langle x|E\rangle\langle E|\phi\rangle \end{aligned} \quad (18a)$$

where $|i\rangle$, $i = 1, 2, \dots, n, \dots$, is some discrete basis system, where the wave functions $\psi(x) \equiv \langle x|\psi\rangle$ and $\phi(E) \equiv \langle E|\phi\rangle$ are smooth functions and where the integrals are Riemann. The probability amplitude in the HS formulation is calculated as:

$$\langle\psi|\phi\rangle = \int_{\text{Lebesgue}} dx \psi^*(x)\phi(x) = \int_{\text{Lebesgue}} dE \psi^*(E)\phi(E). \quad (18b)$$

The integrals are Lebesgue and the set of mathematical wave functions, i.e. the classes $\{\psi_1(x), \psi_2(x), \dots\} \in \mathcal{L}^2$ in (18b), also includes elements that contain no $\psi^{\text{smooth}}(x) \in \mathcal{S}$. Since the physicist only deal with smooth energy distributions $|\phi(E)|^2$ or smooth position distributions $|\psi(x)|^2$ due to the capabilities attributed to their apparatuses, it is of no advantage to choose some arbitrary Lebesgue integrable $\phi(E)$ or $\psi(x)$ as

the tool for the calculation of (18b). A much “more practical method of computation” and the method that a physicist always uses is to choose for the $\phi(E) \in \{\phi(E)\}$ the $\phi(E) = \phi^{\text{smooth}}(E) \in \mathcal{S}$. Then the Lebesgue integrals in (18b) become Riemann integrals and are identical with (18a)⁹. The other integrals in (18b), for which $\{\psi_1(x), \psi_2(x), \dots\}$ does not contain a $\psi^{\text{smooth}}(x)$, have no observational counterpart anyway and are therefore physically useless. They have to be included in (18b) because only with them is the HS a topologically complete space.

Only for $\langle E|\phi\rangle = \phi^{\text{smooth}}(E) \in \mathcal{S}$, and only if one restricts oneself to the integrals of (18a), can one give to the symbol $\langle E|$ in $\langle E|\phi\rangle$ a separate mathematical meaning by interpreting $\langle E|\phi\rangle$ as the value of an anti-linear functional $\langle E|$ at the element $\phi \in \Phi$. This means one can define $\langle E|$ as an element of Φ^\times , the space of continuous anti-linear functionals over Φ . Since this is only possible for the smooth $\phi(E)$, i.e. for the $\phi \in \Phi$, not for all elements of the HS, we obtain for the set Φ^\times a larger set than the set of anti-linear continuous functionals over \mathcal{H} , since from $\Phi \subset \mathcal{H}$ it follows $\mathcal{H}^\times \subset \Phi^\times$. Using the Frechet-Riesz theorem [21] we then identify $\mathcal{H} = \mathcal{H}^\times$ and obtain a triplet of spaces, the Gelfand triplet or rigged Hilbert space (RHS):

$$\Phi \subset \mathcal{H} = \mathcal{H}^\times \subset \Phi^\times, \quad (19)$$

where $|E\rangle \in \Phi^\times$.

The desire for a “more practical method of computation” using Riemann integrals and observable quantities only has thus led to a choice of $\langle E|\phi\rangle$ which allowed us to give a mathematical meaning to the kets $|E\rangle$ and $|x\rangle$. These kets lie outside the HS, whose elements (and therewith also the $\phi \in \Phi \subset \mathcal{H}$) describe ensembles of microphysical systems. These new vectors, the kets $|E\rangle$, are available for a physical interpretation that goes beyond the ensemble interpretation.

Though one can only observe the probabilities (18a) measured with macroscopic apparatuses, it is intuitively attractive to imagine that the effect which the preparation apparatus causes on the registration apparatus is carried from one to the other by some physical entities. These objects, by which the preparation apparatus acts on the registration apparatus (i.e. the carriers of the action), are imagined to be the microphysical systems. Though one cannot see them directly, every physicist believes in them, e.g. believes that the track in a cloud chamber is caused by a single particle. In HS there is nothing that can describe them, but in RHS the kets of Φ^\times may.

Whereas the physical entities connected with an experimental apparatus, like the states ϕ defined by the preparation apparatus or the property ψ defined by the registration

⁹cf. footnote (27) of Sect. (7) for the prerequisites on the operators H, Q to make this possible.

apparatus, are assumed to be elements of \mathcal{H} , or as described above even of Φ , the imagined entities connected with microphysical systems do not have to be in \mathcal{H} or Φ because the energy distribution for a microphysical system does *not* have to be a measurable or even a continuous, infinitely differentiable, rapidly decreasing function of the physical values of E , like the $\langle E|\psi\rangle$ describing the energy resolution of the detector, or the $\langle E|\phi\rangle$ describing the energy distribution of the beam. For the hypothetical entities connected with single microphysical systems one can use Dirac's kets $|E_0\rangle$ with the energy distribution $\langle E|E_0\rangle \propto \delta(E - E_0)$. The energy eigenkets $|E, \Omega\rangle$ (or momentum kets $|\mathbf{p}\rangle$, $E = \mathbf{p}^2/2m, \Omega = \mathbf{p}/|\mathbf{p}|$) represent then the microphysical states of momentum \mathbf{p} . These are not states one can prepare with a macroscopic apparatus, but something that the physicist imagines as single microsystems, something that can be associated with a cloud (or bubble) chamber track.

Summarizing, the HS is too big if one only admits quantities associated with macroscopic apparatuses because apparatuses have smooth energy distributions, $\phi(E) \in \mathcal{S}$ and not every element of $L^2(\mathbb{R})$ describes an energy distribution of an apparatus, i.e., a physically preparable state $\phi \in \Phi$. However, if one also wants to describe single microphysical systems, then the HS is too small, because microphysical states like Dirac's scattering states $|\mathbf{p}\rangle$ cannot be represented in \mathcal{H} . Neither can Gamow's decaying states $|E - i\Gamma/2\rangle$ representing microphysical systems with well defined resonance energy E and lifetime \hbar/Γ be represented in \mathcal{H} . The Gamow kets are the principal objects of this paper.

Usually von Neumann's formulation of quantum mechanics entails some further idealizations in addition to the choice of the HS which one may or may not want to make, like:

- a The one-to-one correspondence between the set of states (equivalence classes of preparation apparatuses) and the set of statistical operators W in \mathcal{H} , or the one-to-one correspondence between pure states and elements of \mathcal{H} . This is already an exaggeration on the pre-Hilbert space level, since not every finite superposition of physical state vectors which will represent a physical state as has recently been emphasized in the discussion of decoherence [22].
- b The one-to-one correspondence between the original set of observables and the set of self-adjoint operators A in \mathcal{H} .
- c The axioms of the idealized measurements (collapse of the wave function). Realistic experiments usually do not even attempt to fulfill the condition of an ideal measurement (of first or second kind) and according to present views this postulate is not needed, since the Schrödinger equation and a measurement scattering process fully describe the measuring act [7, 23].

HS formulation of theoretical quantities	Experimental quantities	RHS formulation of theoretical quantities
Preparation Apparatus defines the		
Density operator W in \mathcal{H}	Physical states (mixtures)	Density operator W in Φ_-
ϕ element of \mathcal{H} or projection operator $ \phi\rangle\langle\phi $	pure states	ϕ element of $\Phi_- \subset \mathcal{H}$
Registration Apparatus defines the		
Unbounded linear operator A in \mathcal{H}	Physical observables	Element of an algebra of τ_Φ -continuous operators
Projection operator Λ or self-adjoint operators F with $0 \leq F \leq 1$ in \mathcal{H}		Projection operators or positive operators in Φ_+
$ \psi\rangle\langle\psi $ one dim. projector in \mathcal{H}	Yes-no observables (property or proposition)	$ \psi\rangle\langle\psi $ with $\psi \in \Phi_+ \subset \Phi$

The spaces Φ_+ and Φ_- , with $\Phi = \Phi_- + \Phi_+$; $\Phi_- \cap \Phi_+ \neq 0$, will be defined in Sect. 5.

Table 1: Comparison of HS and RHS description of physical quantities.

Since these additional idealizations are not of direct relevance for our discussion of quantum mechanical irreversibility and have no bearing on the choice of the most suitable topology for the scalar product space, we shall not discuss them any further here.

After having discussed the basic features of the HS formulation we shall now discuss the RHS formulation of quantum mechanics [24, 25, 26], which has only become possible after the new mathematics of distributions and linear topological spaces had been introduced 30 years after von Neumann's HS.

The RHS formulation is also a mathematical idealization of the structure that one can deduce from physical observations. This idealization provides the fundamental properties (13) and (14) required by Dirac's formalism. It restricts the allowable vectors for the description of entities defined by the experimental apparatus to the subspace $\Phi \subset \mathcal{H}$; Φ is complete with respect to a different topology (meaning of convergence) than \mathcal{H} . But this is of no importance to physics because physics will not use all elements of Φ ; important is that this topology is such that (13) can be mathematically defined (see Sect. 5 below) and (14) can be proved.

The space Φ is specific to the particular quantum physical system considered, and the topology of the space Φ is defined such that:

1. The algebra of observables of the quantum physical system is an algebra of continuous operators in Φ .
2. The Dirac basis vector expansion (14) holds as a theorem, the Nuclear Spectral Theorem.

The distinction between the RHS and the HS formulations of quantum mechanics is summarized in Table I.

For the quantum mechanics of stationary states and reversible processes (using no more than Dirac kets), the HS formulation and the RHS formulation lead to experimental predictions which are only imperceptibly different. The calculations can be written in terms of elements which can be defined with either the HS or the RHS. The standard HS quantum mechanics is just an approximate sub-theory of RHS quantum mechanics. This similarity reflects the similarity of von Neumann's HS formulation and Dirac's incomplete bra and ket formalism. The RHS formulation of quantum mechanics makes the Dirac formalism rigorous and provides a more "practical method of computation" in the domain where both theories overlap, e.g. by allowing the use of Riemann integrals in computations like (18) rather than Lebesgue integrals.

But with the new mathematical language that the RHS provides one can speak and think new physics. The microphysical scattering states (hyperbolic orbits of the corresponding classical system) described by the Dirac kets $|E\rangle$ gave already an indication of this new domain. The really new physics of the RHS formulation is the microphysical irreversibility which is exemplified by the semigroup time evolution of the Gamow kets which represent decaying states or resonance states. This irreversible semigroup time evolution was unthinkable in the old HS formulation, where decaying states had complicated, problematic features, as we shall discuss next.

4 Consequences of the HS Formulation and Some Views on Time-Asymmetry in Quantum Mechanics

Most computations in quantum theory do not use the completeness property of the HS; they work only with properties of a pre-Hilbert space. But there are some general results that one obtains from HS-completeness which unveil the problems of the HS formulation of quantum decay:

1. There is no vector that obeys the exact exponential decay law [27]. Mathematically stated, there is no $\phi \in \mathcal{H}$ whose survival probability

$$\mathcal{P}_s \equiv |(\phi, e^{-iHt}\phi)|^2$$

has the property

$$\mathcal{P}_s = e^{-\Gamma t}. \quad (20)$$

Although this “deviation from the exponential law” is unobserved, the theoretical prediction of it has led some to infer that exact exponential decay does not exist in nature, instead of suspecting a flaw in the mathematical idealization of the HS. Since magnitudes cannot be predicted from only mathematical properties, one can argue that this deviation from the exponential law is smaller than any experimental error. Thus these deviations could always be smaller than what can be experimentally ruled out and therefore cannot be tested¹⁰. In spite of the untestability of these mathematical deviations, alternate explanations for the observed exponentiality have been proposed. For example, the observed exponential behavior has been ascribed to the influence of the environment, to the measurement process or to both [28].

Physicists usually demand even more of a decaying state than (20). As envisioned by Gamow [29], physicists would like to describe a decaying state by an eigenvector $\psi^G \equiv |E_R - i\Gamma/2\rangle$ of the self-adjoint Hamiltonian H with complex eigenvalue $(E_R - i\Gamma/2)$, i.e.

$$H|E_R - i\Gamma/2\rangle = (E_R - i\Gamma/2)|E_R - i\Gamma/2\rangle, \quad (21)$$

and the exponential time evolution

$$\psi^G(t) = e^{-iHt}\psi^G = e^{-iE_R t}e^{-\frac{\Gamma}{2}t}\psi^G \quad (22)$$

but such vectors do not exist in the HS.

Empirically, stable particles are not considered qualitatively different from quasi-stable particles, but are only quantitatively different by a zero or very small value of Γ . A particle decays if it can decay and it is stable if selection rules for some quantum numbers prevent it from decaying. Both stable and decaying states have been described in elementary textbooks [30], in successful phenomenological or effective theories [31] and in tables of experimental data [32], as autonomous entities characterized by E_R and Γ , where Γ can sometimes be equal to zero. A vector like ψ^G fulfilling (21) and (22) would have the suitable properties.

Though there are no ψ^G in the HS, the RHS contains these Gamow vectors ψ^G . They are in a conjugate space Φ_+^\times of a RHS, $\Phi_+ \subset \mathcal{H} \subset \Phi_+^\times$.

¹⁰Probabilities like \mathcal{P}_s are always observed as ratios of (preferably large) integers $N(t)/N(0)$ and not as real numbers like $e^{-\Gamma t}$, therefore any discussion of deviations from the exponential law is futile as long as the deviations are not predicted with a magnitude that can be compared with $1/\Gamma$.

2. Decay probabilities in the HS theory are identically zero. The probability for the transition from a state $\psi(t) = U^\dagger(t)\psi = \exp(-iHt)\psi$ into the decay products described by the subspace $\Lambda\mathcal{H} \subset \mathcal{H}$, where Λ is the projection operator on the subspace of decay products (or Λ is a positive operator), is in quantum theory given by

$$\mathcal{P}(t) = \text{Tr}(\Lambda|\psi(t)\rangle\langle\psi(t)|) = \langle\psi|e^{iHt}\Lambda e^{-iHt}|\psi\rangle. \quad (23)$$

This is the probability to detect the observable Λ in the state $|\psi(t)\rangle\langle\psi(t)|$. The Hamiltonian H is always assumed to be self-adjoint and semi-bounded $H \geq 0$, (the condition for the stability of matter). The decay of a prepared quasi-stationary state is assumed to start at finite time, $t > t_2 > -\infty$, which is mathematically formulated as:

$$\int_{t_1}^{t_2} \langle\psi(t)|\Lambda|\psi(t)\rangle dt = 0 \quad (24)$$

for some t_1 such that $-\infty \leq t_1 < t_2$ (t_2 is usually chosen to be $t_2 = 0$). With these assumptions one can show [33]

$$\mathcal{P}(t) \equiv 0 \text{ for all } t \text{ (precisely, almost all } t) \text{ and for every } \psi \in \mathcal{H}. \quad (25)$$

This means that in HS the transition probability $\mathcal{P}(t) = \text{Tr}(\Lambda(t)\rho)$ of any state $\rho = \sum_i w_i |\psi_i\rangle\langle\psi_i|$ is identically zero if the transition starts at a finite time. In particular there are no decaying states in \mathcal{H} . This mathematical result is not usually interpreted as a deficiency of the HS idealization but as being due to some problems with causality [34].

The way out of this dilemma is shown in Sect. 7.4. One replaces ψ of the HS in (23) by the Gamow vector ψ^G of the RHS, which is not in the HS. Then the transition probability $\mathcal{P}(t)$ can be shown to be nonzero and exponentially approaching unity, i.e., $\mathcal{P}(t) \propto [1 - \exp(-\Gamma t)]$, for $t > 0$.

3. In the HS formulation of quantum mechanics the symmetry transformations (e.g. Galilean transformations, Poincaré transformations) are described by a unitary, projective group representation in \mathcal{H} . This implies that the time evolution is unitary and reversible and given by $U^\dagger(t) = \exp(-iHt)$, $-\infty < t < \infty$.

Instead of recognizing that this may be a property of the mathematical idealization imposed by the HS, the widespread conclusion was that quantum mechanical irreversibility of isolated microphysical systems is impossible.

Lately, there have been several exceptions to the widespread conclusion that the time evolution described by the Hamiltonian must be time-reversible. Different people mentioned different reasons why a microphysical “arrow of time” should exist.

- i Peierls [35] and his school emphasized the importance of the initial and boundary conditions. They chose purely outgoing boundary conditions for the solutions of the Schrödinger equation. According to Peierls' ideas, irreversibility is connected with the choice of the boundary or initial conditions (for the “wave functions”).
- ii T. D. Lee [36] explained that the time reverse of a decay process is impossible or highly improbable to attain due to the phase of the state vector of the quantum system. These arguments immediately extend to scattering experiments.

It is easy to prepare two uncorrelated incoming beams that scatter into strongly correlated outgoing spherical waves as done in a typical scattering experiment. It is experimentally hopeless to prepare a state consisting of two strongly correlated spherical waves (with fixed relative phase) in such a way that after the scattering of two uncorrelated plane waves emerge. The latter would be the time reverse of the setup for a typical scattering experiment.

- iii Ludwig [7] had also noticed that in an experiment with quantum systems one could not time translate the trigger of a detector that registers an observable to a time before the preparation apparatus (e.g. accelerator) has been turned on. This preparation \longrightarrow registration arrow of time for the apparatuses ought to be transcribed into a semigroup evolution of the state (defined by the preparation apparatus) relative to the observables (defined by the detector)¹¹. However, knowing that in HS the time evolution is represented by the unitary group, he extrapolated this semigroup to all times $-\infty < t < \infty$.
- iv The semigroup generated by the Hamiltonian emerged unexpectedly in the mathematical derivation of the time evolution for the Gamow vectors (22) and therewith introduced microphysical irreversibility into the RHS formulation of quantum mechanics [37].
- v Prigogine and his school had emphasized for a long time [38] that irreversibility is fundamental. It therefore should be intrinsic to the dynamics of the microsystems rather than being caused by external effects of a quantum reservoir or the environment (measurement apparatus). Irreversibility therefore should be connected with the Hamiltonian (or Liouvillian) and described by a non-unitary transformation.
- vi I. Antoniou [39] showed that the formalism of subdynamics leads to the RHS and suggested for this non-unitary transformation the semigroup of the Gamow vectors generated by the extended Hamiltonian in the RHS.

¹¹cf. Sect. 6, footnote (15).

- vii Coming from an entirely different problem, Gell-Mann and Hartle [8] introduced an arrow of time in the quantum mechanics of cosmology. In order to avoid inconsistencies for the probabilities of histories, they required a time ordering for the projection operators in a history and the initial states (cf. (5) and (6) of Preface). This arrow of time is identical to the preparation→registration arrow of time if one applies the same formula (viz. (5) Preface) to the probability to register the observable (projection operator) in the prepared state ρ of a laboratory experiment.

All the examples above are different manifestations of a fundamental quantum mechanical distinction between the past and the future—different expression of causality in quantum physics. The paradigm of this microphysical irreversibility is the time evolution of resonances. Since resonances abound, this time-asymmetry is prevalent. HS quantum theory cannot describe it, however the same mathematical theory of the rigged Hilbert space that was introduced to justify the Dirac formalism also describes the irreversible decay of microsystems and allows for the mathematical transcription of the quantum mechanical arrow of time. It describes irreversibility on the microphysical level.

5 From a Pre-Hilbert Space to the Rigged Hilbert Space

“A role of rigorous mathematics in physical science is to make sense of heuristic ideas (i.e. find the ‘correct setting’)—not to assert they are nonsense.”

—M. Fisher¹²

A pre-Hilbert space is a linear space Ψ with a scalar product. This scalar product is denoted by

$$(\psi, F) \text{ or by } \langle \psi | F \rangle. \quad (26)$$

The pre-Hilbert space is without any topological structure¹³. Such mathematical concepts as neighborhoods, convergence of infinite sequences, topological completeness, continuous operators, continuous functionals, dense subspaces, etc. are not defined. This pre-Hilbert space Ψ is what one usually means in physics when one speaks of the Hilbert space, and this is what one mostly uses, together with the representations of ψ by square integrable functions and the calculation of scalar products as Riemann integrals. The full mathematical

¹²From talk “What’s Mathematical Physics to Physicists? Some examples from Past, Present and Future” by M. Fisher, VIIth International Congress on Mathematical Physics, Boulder, Colorado, August 1983.

¹³cf. footnote (5).

structure of the Hilbert space \mathcal{H} is much more complicated, and the proof of statements like the deviation from the exponential decay law in Sect. 4.1 and the vanishing of the transition probability in Sect. 4.2 require the precise mathematical definition of \mathcal{H} .

The linear scalar product space Ψ can be endowed with various topologies, which means that various definitions for the convergence of infinite sequences can be given. We denote these topologies by $\tau_{\mathcal{H}}$ for the HS, for example, and label the topological notions with it, such as $\tau_{\mathcal{H}}$ -convergence. A space is completed by adjoining to it the limit elements of all Cauchy sequences not already contained in the space.

The Hilbert space \mathcal{H} is the completion of the pre-Hilbert space Ψ with respect to the convergence defined by the norm $\|\phi\| = \sqrt{(\phi, \phi)}$. If one represents the space Ψ by a space of functions and the scalar product by the usual integral, then one cannot complete the space Ψ into a Hilbert space \mathcal{H} unless the integral is a Lebesgue integral (as opposed to the more frequently used Riemann integral). In this case, each element $\phi \in \mathcal{H}$ is represented by a *class* of Lebesgue square-integrable functions which differ on a set of Lebesgue measure zero (e.g. all rational numbers), not by one wave function. This is the “realization” of \mathcal{H} by $L^2(\mathbb{R})$.

The RHS is the same linear space Ψ completed with respect to three different topologies: one stronger, one the same, and one weaker than the topology defined by the Hilbert space norm. The stronger topology τ_{Φ} , e.g. a topology defined by a countable number of norms, leads by completion to the space Φ , which, as a consequence of its stronger topology, has the property $\Phi \subset \mathcal{H}$. The topological dual to Φ , i.e. the space of *continuous* anti-linear functionals on Φ , is denoted by Φ^{\times} and since its topology is weaker than the Hilbert space topology we have $\mathcal{H} \subset \Phi^{\times}$. The antilinear τ_{Φ} -continuous functionals or functions $F(\phi)$ over the space of $\phi \in \Phi$ are denoted by $F(\phi) = \langle \phi | F \rangle$. This bra-ket is an extension of the scalar product (ϕ, f) , $f \in \mathcal{H}$ to those elements $F \in \Phi^{\times}$ which are not elements of \mathcal{H} . By completing Ψ with respect to these three topologies, one obtains a triplet of spaces. This is the Gelfand triplet or rigged Hilbert space [18].

$$\Phi \subset \mathcal{H} = \mathcal{H}^{\times} \subset \Phi^{\times}, \quad (27)$$

$$\begin{aligned} \text{with elements “bra” and “ket” } & \langle \phi | \in \Phi \quad |F\rangle \in \Phi^{\times} \\ \text{or “ket” and “bra” } & |\phi\rangle \in \Phi \quad \langle F| \in \Phi^{\times} \end{aligned} \quad (28)$$

The vectors $\phi \in \Phi$, either as a ket $|\phi\rangle$ or bra $\langle \phi|$, represent physical quantities connected with the experimental apparatuses. For example, in a scattering experiment the preparation apparatus (accelerator) defines the state ϕ and the observable $|\psi\rangle\langle\psi|$ is defined by the registration apparatus (detector), and both ϕ and ψ are elements of Φ . The vectors $|F\rangle$

or $\langle F| \in \Phi^\times$ represent quantities connected with the microphysical system, e.g. scattering states $|E\rangle$ or decaying states $|E_R - i\Gamma/2\rangle$.

The general observable is now represented by a continuous (also bounded) operator A in Φ (but in general by an unbounded operator \overline{A} or A^\dagger in \mathcal{H}) and one now has a triplet of operators corresponding to the triplet of spaces (27)

$$A^\dagger|_\Phi \subset A^\dagger \subset A^\times. \quad (29)$$

The operator A^\dagger is the Hilbert space adjoint of A (if A is essentially self-adjoint, then $A^\dagger = \overline{A}$, where \overline{A} denotes the closure of the operator A in \mathcal{H}). The operator $A^\dagger|_\Phi$ is its restriction to the space Φ , and the operator A^\times in Φ^\times is the conjugate operator of A defined by

$$\langle A\phi|F\rangle = \langle \phi|A^\times F\rangle \text{ for all } \phi \in \Phi \text{ and all } |F\rangle \in \Phi^\times. \quad (30)$$

By this definition, A^\times is the extension of the operator A^\dagger to the space Φ^\times and not the extension of the operator A , which is more often used in mathematics. The operator A^\times is *only defined* for an operator A which is *continuous* (and bounded) in Φ , so A^\times is always a continuous operator in Φ^\times . In quantum mechanics it is empirically impossible to restrict oneself to continuous (and therefore bounded) operators \overline{A} in \mathcal{H} (e.g. already the observables in $[P, Q] = -i1$ cannot be represented by $\tau_{\mathcal{H}}$ -continuous operators in \mathcal{H} , but they can be represented by τ_Φ -continuous operators in a space Φ with stronger topology). But one can restrict oneself to algebras of observables $\{A, B, \dots\}$ described by continuous operators in Φ if the topology of Φ is suitably chosen. Then $\{A^\times, B^\times, \dots\}$ are defined and continuous in Φ^\times . If A in (30) is not self-adjoint then $A^\dagger|_\Phi$ need not be a continuous operator in Φ even if A is, but one can still define the conjugate A^\times using (30), which is a continuous operator in Φ^\times .

A generalized eigenvector of an operator A is defined to be that $F \in \Phi^\times$ which fulfills

$$\langle A\phi|F\rangle = \langle \phi|A^\times F\rangle = \omega \langle \phi|F\rangle \text{ for all } \phi \in \Phi, \quad (31)$$

where the complex number ω is called the generalized eigenvalue. This is also written as

$$A^\times|F\rangle = \omega|F\rangle. \quad (32)$$

For an essentially self-adjoint operator ($A^\dagger = \overline{A}$ = closure of A) this is often also written as it is in Dirac's formalism, $A|F\rangle = \omega|F\rangle$, especially if one avoids mentioning the topological structure and works with just a linear scalar product space Ψ . The precise meaning of eigenvalue equations like (13),(21) and (22) is given by (31).

The generalized eigenvalue ω in (31) may belong to the continuous spectrum of the Hilbert space operator A^\dagger , as is usually assumed for Dirac kets, but it need not belong to

the Hilbert space spectrum of A^\dagger , and for a self-adjoint operator A^\dagger (i.e. if A is essentially self-adjoint) ω need not even be real. If $|F\rangle \in \mathcal{H}$, then (31) is identical to the definition of an ordinary eigenvector of A^\dagger in \mathcal{H} with discrete eigenvalue ω .

The possible set of generalized eigenvalues of an operator A is determined by the choice of the space Φ^\times , or equivalently, by the choice of Φ (i.e. by the topology τ_Φ with which we choose to equip the linear space Ψ). One can choose the topology τ_Φ with respect to which one completes the pre-Hilbert space Ψ unwisely so that Φ^\times contains all kinds of things. Others have suggested restricting τ_Φ such that the set of generalized eigenvalues defined by (31) is identical with the Hilbert space spectrum of A^\dagger [40]. This would reproduce the Dirac formalism to the extent that it has been used in the past, because the Dirac kets are always chosen such that they are connected with the (absolutely) continuous spectrum of an operator $A^\dagger = \bar{A}$. But this would not allow for Gamow kets like those in (21), and no new physics would be incorporated in such a “tight rigging”.

We invoke the principle that the space Φ (and its dual Φ^\times) should be chosen by physical arguments. For one “kind” of quantum physical system (where the term “kind” is left to the intuitive interpretation based on experimental experience) one takes as a mathematical image one particular space Φ . For instance, one could define the topology for Φ such that the observables of the physical system under consideration are continuous operators in Φ [25, 26]. We shall discuss a scattering experiment and additional physical arguments related to causality and initial conditions which will be used to specify Φ further. Causality and the choice of initial conditions are old principles of classical physics which have not been fully utilized in standard quantum mechanics.

6 Time-Symmetric Equations and Time-Asymmetric Boundary Conditions

“The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve.”

—E.P. Wigner¹⁴

Theoretical predictions are based on two prerequisites [41]: the *laws of nature* and the *initial (or boundary) conditions*. The laws of nature provide the observables, e.g. the algebra of operators derived from a space-time symmetry group or the Hamiltonian H and the dynamical equations, which are the same in both the HS and RHS formulations and given

¹⁴ E.P. Wigner, *Symmetries and Reflections*, (Ox Bow Press, Woodbridge, Connecticut, 1979), p. 237.

by the Schrödinger equation or the von Neumann equation:

$$i\hbar \frac{d\phi(t)}{dt} = H\phi(t) \quad \text{or} \quad i\hbar \frac{dW(t)}{dt} = [H, W(t)]. \quad (33)$$

The laws of nature are not subject to human influences; they are given once and forever. The initial and boundary conditions leave some freedom of choice but are limited by the achievability—in principle and in practice—of building experimental apparatuses and by causality [35].

In the standard HS formulation of quantum mechanics, one chooses the initial and boundary conditions for the $|\phi\rangle$ and the $|F\rangle$ such that both are always elements in \mathcal{H} . In this way one does not take into consideration the variety of possible choices and limitations for the initial and boundary conditions due to causality. In the RHS formulation, the initial condition can be chosen more specific to the particular problem under consideration. We shall use these additional conditions to define the space Φ , or equivalently the RHS $\Phi \subset \mathcal{H} \subset \Phi^\times$, for a quantum scattering system.

The best known example of a RHS is the space in which Φ is realized by the Schwartz space of “well-behaved” functions, i.e. functions that have derivatives which are all continuous, smooth and rapidly decreasing, \mathcal{H} is the space of Lebesgue square integrable functions, and Φ^\times is the space of tempered distributions, $\mathcal{S} \subset L^2(\mathbb{R}) \subset \mathcal{S}^\times$. This is the RHS suitable for the quantum oscillator whose algebra of observables is represented by τ_Φ -continuous operators.

In quantum theory, if one distinguishes between preparations and registrations¹⁵, one can further specify the RHS, and one is led to a pair of RHS’s: one representing the preparations and the other representing the registrations. As an example we will discuss a scattering experiment. The scattering experiment can be subdivided into a preparation stage and a registration stage. The in-state ϕ^+ that evolves from the prepared in-state ϕ^{in} outside the interaction region is determined by the preparation apparatus (the accelerator). The “out-state” ψ^- , detected as the “out-state” ψ^{out} outside the interaction region, is determined by the registration apparatus (the detector). According to the physical interpretation of the RHS formulation, real physical entities connected with the experimental apparatuses, e.g. the ensemble $|\phi\rangle\langle\phi|$ describing the preparation apparatus or the observable $|\psi\rangle\langle\psi|$ describing the registration apparatus are described by the *well-behaved* vectors $\phi, \psi \in \Phi$. However, states

¹⁵ The one feature on which most discussions of the foundations of quantum mechanics agree is the dichotomy of “state” and “observable”. If one interprets quantum theory objectively from the outside in terms of classical physics, as in [7], then the state is defined by the preparation apparatus (e.g. accelerator) and the observable is defined by the registration apparatus (detector). Though distinct by their definition, the HS formulation blurs this differentiation between states and observables by not specifying which elements of the HS are allowed as “states” and which as “observables”.

are different from observables and should be described by a different set of vectors. We denote the space of state vectors ϕ^+ by Φ_- and the space of observable vectors ψ^- by Φ_+ , where $\Phi = \Phi_- + \Phi_+$ and $\Phi_- \cap \Phi_+ \neq 0$. We will call the elements of Φ_- and Φ_+ *very well-behaved* vectors from below and above, respectively, where below and above refer to the second sheet of the energy surface of the analytically continued S -matrix. In place of the single rigged Hilbert space we therefore have a pair of rigged Hilbert spaces:

$$\phi^+ \in \Phi_- \subset \mathcal{H} \subset \Phi_-^\times \quad \text{for ensembles or prepared in-states,} \quad (34)$$

$$\psi^- \in \Phi_+ \subset \mathcal{H} \subset \Phi_+^\times \quad \text{for observables or registered “out-states”.} \quad (35)$$

Here the Hilbert space \mathcal{H} is the same for both triplets, and since we ignore the discrete eigenvalues here it is $\mathcal{H}^{\text{a.c.}}$ of the continuous spectrum.

Mathematically, these spaces can be defined by their realizations as function space as in the case of the Schwartz space triplet $\mathcal{S} \subset L^2(\mathbb{R}) \subset \mathcal{S}^\times$ above. The spaces Φ_+ and Φ_- are defined as the spaces that are realized by the well-behaved Hardy class functions [42] in the upper half plane, $\mathcal{S} \cap \mathcal{H}_+^2|_{\mathbb{R}^+}$, and in the lower half plane, $\mathcal{S} \cap \mathcal{H}_-^2|_{\mathbb{R}^+}$, respectively:

$$\phi^+ \in \Phi_- \quad \text{if and only if} \quad \langle^+ E | \phi^+ \rangle \in \mathcal{S} \cap \mathcal{H}_-^2|_{\mathbb{R}^+} \quad (36)$$

$$\psi^- \in \Phi_+ \quad \text{if and only if} \quad \langle^- E | \psi^- \rangle \in \mathcal{S} \cap \mathcal{H}_+^2|_{\mathbb{R}^+} \quad (37)$$

The notation $|_{\mathbb{R}^+}$ mean restriction to the positive real line, i.e. the physical values of energy. If \mathcal{H} in (34) and (35) denotes the Hilbert space realized as the space of Lebesgue square integrable functions, $L^2[0, \infty) = L^2(\mathbb{R}^+)$, then one can show [43] that

$$\mathcal{S} \cap \mathcal{H}_\pm^2|_{\mathbb{R}^+} \subset L^2(\mathbb{R}^+) \subset (\mathcal{S} \cap \mathcal{H}_\pm^2|_{\mathbb{R}^+})^\times \quad (38)$$

are two rigged Hilbert spaces of functions. The two RHS's of the in-states $\{\phi^+\}$ (34) and of the out-states $\{\psi^-\}$ (35) are mathematically defined as those RHS's whose realizations are the two RHS's of $\mathcal{S} \cap \mathcal{H}_-^2|_{\mathbb{R}^+}$ and $\mathcal{S} \cap \mathcal{H}_+^2|_{\mathbb{R}^+}$, respectively.

The opposite sub- and superscripts for vectors and spaces comes from the opposite nomenclature in physics (scattering theory for ϕ^+, ψ^-) and mathematics (theory of Hardy class functions for $\Phi_- \doteq \mathcal{S} \cap \mathcal{H}_-^2|_{\mathbb{R}^+}$ and $\Phi_+ \doteq \mathcal{S} \cap \mathcal{H}_+^2|_{\mathbb{R}^+}$) which had been developed independently. Except for the nomenclature the spaces are the “same”, i.e. $\{\phi^+\} = \Phi_-$ and $\{\psi^-\} = \Phi_+$. This coincidence is another example of “The Unreasonable Effectiveness of Mathematics in Natural Sciences”¹⁶.

It is of course not obvious at all that the physical spaces (34) and (35), the mathematical images of the prepared states and the detected observables respectively, should have anything

¹⁶cf. footnote (14).

to do with the Hardy class spaces (38). Originally the Hardy class property was introduced in order to derive (21) as a generalized eigenvalue equation and obtain a Breit-Wigner energy distribution for the vector associated with the resonance pole of the S -matrix [37, 44]. But the Hardy class property of the in-states $\phi^+ \in \Phi_-$ and $\psi^- \in \Phi_+$ is much more generally valid and can be obtained as a consequence of the Paley-Wiener theorem [42] from a mathematical formulation of causality [45].

The condition of causality can be formulated without any reference to the mathematical theory of quantum mechanics. Therefore this formulation of causality does not depend on the choice of the HS or RHS formulation. It is phrased in terms of only the intuitive properties of the macroscopic preparation and registration apparatuses. We call this theory-independent statement of causality the preparation \rightarrow registration arrow of time¹⁷:

Let $t = t_0 = 0$ denote the point in time before which the preparation is complete and after which the registration begins. Then:

1. Time translation of the registration apparatus relative to the preparation apparatus makes sense only by an amount $t \geq 0$.

A second version of this statement uses quantum theoretical notions; we call it the quantum mechanical arrow of time.

2. (Heisenberg picture) An observable $|\psi^-(t)\rangle\langle\psi^-(t)|$ can be measured in a state $\phi^+ (= \phi^+(0))$ only after the state has been prepared, i.e., for $t \geq 0$.

or

2. (Schrödinger picture) A state $\phi^+(t)$ must be prepared before an observable $|\psi^-\rangle\langle\psi^-| (= |\psi^-(0)\rangle\langle\psi^-(0)|)$ can be measured in that state, i.e., ϕ^+ must be prepared during $t \leq 0$.

This has implications for the most fundamental quantities in quantum physics, the expectation values. These probabilities $\mathcal{P}(t)$ are observed experimentally as a ratio of (large) numbers (detector counts) $N(t)/N = \mathcal{P}(t)$ and calculated theoretically as

$$\mathcal{P}(t) = \text{Tr}(\Lambda(t)W) = \text{Tr}(\Lambda W(t)), \text{ where} \quad (39a)$$

$$\Lambda(t) = e^{iHt}\Lambda e^{-iHt} \text{ and } W(t) = e^{-iHt}W e^{iHt}. \quad (39b)$$

where Λ denotes the observable and W the state.

¹⁷It is curious that the theory-independent version (Version 1) of this arrow of time had been known for some time [7], but was then extrapolated “into the past” in order to obtain the unitary one-parameter group $U(t) = e^{iHt}$ for the time translation of the observables in the HS.

In many cases, such as for stationary states and time-independent observables, the right hand side of (39a) can be calculated for any value of t , i.e. the equation is valid for $-\infty < t < \infty$. The operators act in the space $\mathcal{H}^{\text{disc}}$ of the discrete spectrum and all calculations are the same in the RHS formulation and the HS formulation. The physics of stationary states is not affected by the arrow of time.

This does not hold generally. In a scattering experiment, for example, the state has to be prepared. If $t = t_0 = 0$ is the time at which the preparation of the state $W = |\phi^+\rangle\langle\phi^+|$ is complete and the registration of $\Lambda(t) \equiv |\psi^-(t)\rangle\langle\psi^-(t)|$ can begin, then

$$\frac{N(t)}{N} = \mathcal{P}(t) = \text{Tr}(\Lambda W(t)) = \text{Tr}(\Lambda(t)W) = |\langle\psi^-(t)|\phi^+\rangle|^2, \text{ for } t \geq 0 \text{ only.} \quad (40)$$

The left hand side, $N(t)/N$ is observed only for $t \geq 0$ and assumed to be zero for $t < 0$ (detector clicks before $t = 0$ are discounted as noise):

$$\frac{N(t)}{N} = \mathcal{P}(t) = 0, \text{ for } t < 0. \quad (41)$$

In the HS, $W(t)$ of (39b) is in all cases calculated for positive as well as negative t . The RHS formulation allows for $W(t)$'s which can only be *calculated* for $t \geq 0$, e.g. the Gamow states $W(t) = |\psi^G(t)\rangle\langle\psi^G(t)|$ of Sect. 7.2. If the time scale of the formation of $W(t)$ around $t_0 = 0$ is orders of magnitude smaller than the time scale of decay it is always more practical to work with (41).

The validity of the preparation—registration arrow (Version 1) is obvious. With the association between the experimental and the theoretical quantities of Table I, Version 2 (Heisenberg picture) should be an obvious consequence of Version 1 if for all times t these associations are upheld.

In the HS formulation this is not possible, let $t = 0$ be the point in time before which the preparation is completed and after which the registration begins. Because in the HS formulation, the time translation operator $U^\dagger(t)$ for the state ϕ (and $U(t)$ for the observable ψ) is defined for $-\infty < t < +\infty$, one can calculate $|\psi(\tau)\rangle\langle\psi(\tau)| = U(\tau)|\psi\rangle\langle\psi|U^\dagger(\tau)$ for *any* negative value of τ , and its expectation value in every state $\phi \in \mathcal{H}$ is in general not zero for $\tau < 0$. Thus, in the HS formulation it is possible to calculate a non-zero expectation value for an observable at times $\tau < 0$, which one should not observe according to Version 1.

In the RHS formulation the quantum mechanical arrow of time can be implemented in the following way [45]: the prepared state vectors $\phi^+ \in \Phi_-$ representing the preparation apparatus should have a smooth energy wave function $\langle^+ E|\phi^+\rangle \in \mathcal{S}|_{\mathbb{R}^+}$ whose Fourier transform $\mathcal{F}(\tau) \in \mathcal{S}$ should be zero for $\tau > 0$, because $\tau = 0$ is the time after which there is no preparation. The vector $\psi^- \in \Phi_+$ describing the observable defined by the registration

apparatus should have a smooth energy wave function $\langle^{-}E|\psi^{-}\rangle \in \mathcal{S}|_{\mathbb{R}^+}$ whose Fourier transform $\mathcal{G}(\tau) \in \mathcal{S}$ is zero for $\tau < 0$ because $\tau = 0$ is the time until the registration apparatus remains turned off. From this property of the Fourier transforms it follows by the use of the Paley-Wiener Theorem that the wave functions must have the properties

$$\langle^{+}E|\phi^{+}\rangle \in \mathcal{H}_{-}^2 \quad \text{and} \quad \langle^{-}E|\psi^{-}\rangle \in \mathcal{H}_{+}^2$$

This means they have the properties (36) and (37) which leads to the two different spaces Φ_{-} and Φ_{+} (34) and (35) for the states and observables, respectively. The use of different mathematical spaces for states (in-states) and observables (so-called “out-states”) is one of the new features of the RHS formulation of quantum mechanics.

Neither Φ nor \mathcal{H} can describe single microsystems as mentioned above in Sect. 3.3. However, it is intuitively attractive to imagine that the preparation apparatus acts on the registration apparatus by the action of single physical entities, the microphysical systems. The energy distribution for a microphysical system does not have to be a “well-behaved” function of the physical values of energy E . For the entities connected with microphysical systems the RHS formulation provides the elements of Φ^{\times} , Φ_{+}^{\times} and Φ_{-}^{\times} . In particular, Dirac’s scattering states $|\mathbf{p}^{\pm}\rangle$ are elements of Φ_{\mp}^{\times} , Gamow’s decaying states $\psi^G = |E_R - i\Gamma/2^{-}\rangle$ are elements of Φ_{+}^{\times} and (since resonance poles come in pairs at the complex energies $(E_R \mp i\Gamma/2)$) the exponentially growing Gamow vectors $|E_R + i\Gamma/2^{+}\rangle$ are elements of Φ_{-}^{\times} .

7 Consequences of the RHS formulation of Quantum Mechanics

7.1 Semigroup Time Evolution from Time Asymmetry and Some Remarks on Time Reversal Transformations

“When in the 18th century Euler discovered those formulas which till today delight the mathematical phantasy, he seriously stated that his pencil was more clever than himself. This impression that mathematical structures can include a kind of self-determination concerns me at this time. . . . Mathematics and Philosophy attack the world’s problems in different ways. Only by their complementary action do they give the right direction.”

—E. Kähler¹⁸

Once the spaces (34) and (35),

$$\begin{aligned} \Phi_{-} &\subset \mathcal{H} \subset \Phi_{-}^{\times} && \text{for ensembles or prepared in-states—}\{\phi^{+}\} \text{ and} \\ \Phi_{+} &\subset \mathcal{H} \subset \Phi_{+}^{\times} && \text{for observables or registered “out-states”—}\{\psi^{-}\}, \end{aligned}$$

¹⁸Preprint (1996), translated by the authors.

are chosen, which we did in Sect. 6 on the basis of some causality arguments, then it is easy to see mathematically that the extension of the time evolution operator $U^\dagger(t) = e^{-iHt}$ in \mathcal{H} to an operator $U_+^\times(t)$ in Φ_+^\times —the conjugate of the operator $U(t)$ in Φ_+ as defined by (30)—is only a semigroup¹⁹.

Historically, this was not the situation because ingrained in our thinking was the notion that the time evolution operator $U(t) = e^{iHt}$ in quantum mechanics—and the representation of any continuous symmetry transformation—are unitary (reversible) group operators. Philosophizing alone would not be enough to take a semigroup instead. To arrive at the semigroup, we start from the empirically desirable properties (21) and (22) of Gamow resonance states and let mathematics determine the path. Then the result will still be surprising, but now acceptable.

We use the definition of a resonance as a (pair of) first order poles in the analytically continued S -matrix, $S_{bb'}(E) = \langle b|S(E)||b'\rangle$. In the representation of the S -matrix

$$(\psi^{out}, S\phi^{in}) = (\psi^-, \phi^+) = \sum_{bb'} \int_{\text{spectrum of } H} \langle \psi^-|b, E^- \rangle \langle b|S(E)||b'\rangle \langle +b', E|\phi^+ \rangle, \quad (42)$$

we deform the contour of integration from the cut $\{0 \leq E < \infty\}$ (the spectrum of H) into the second sheet of the lower complex plane. The Dirac kets $|E^- \rangle = |b, E^- \rangle$ (where b is the degeneracy quantum numbers) with $E \in \{\text{continuous (real) spectrum}\}$ are therewith continued to complex values. At the position of the resonance pole $E = z_R = E_R - i\Gamma/2$, they become—using Cauchy’s formula—the Gamow kets $|z_R^- \rangle$. This is standard scattering theory except that we want to be careful about the $\langle \psi^-|E^- \rangle$ and the $\langle +E|\phi^+ \rangle$ and do the analytic continuation only for those ψ^- and ϕ^+ for which it is mathematically allowed. This means that $\langle \psi^-|E^- \rangle = \langle E^-|\psi^- \rangle^*$ and $\langle +E|\phi^+ \rangle$ must not only be elements of \mathcal{S} , but also must be the boundary value of an analytic function. We repeat the same process for $\langle H\psi^-|E^- \rangle = \langle \psi^-|H^\times|E^- \rangle = E\langle \psi^-|E^- \rangle$. In order that (21) holds as a general eigenvalue equation,

$$\langle \psi^-|H^\times|z_R^- \rangle = z_R \langle \psi^-|z_R^- \rangle, \quad (43)$$

the $\langle \psi^-|E^- \rangle$ must further be required to be of Hardy class, $\langle \psi^-|E^- \rangle \in (\mathcal{S} \cap \mathcal{H}_-^2)$, in other words, ψ^- must be required to be a Hardy class vector, $\psi^- \in \Phi_+$. Similarly we must require $\phi^+ \in \Phi_-$.

Now, in order to calculate (22) in its mathematically precise form as a generalized eigenvector equation, one repeats the same process for ψ^- in (42) replaced by $\psi^-(t) = e^{iHt}\psi^-$ —the observable $|\psi^- \rangle \langle \psi^-|$ translated by time t . On the right hand side of (42) one then obtains

¹⁹For a semigroup, the inverse $(U^\times(t))^{-1}$ does not have to exist.

$\langle \psi^-(t) | E^- \rangle = \langle \psi^- | e^{iH^\times t} | E^- \rangle = e^{-iEt} \langle \psi^- | E^- \rangle$ in place of $\langle \psi^- | E^- \rangle$. The contour deformations needed on the right hand side of (42) are now possible if and only if $t \geq 0$, because then and only then is $e^{-iEt} \langle \psi^- | E^- \rangle$ a Hardy class function. Then and only then, one obtains, in the same way as (43) was obtained, the following result²⁰:

$$\langle \psi^- | e^{-iH^\times t} | z_R^- \rangle = e^{-iz_R t} \langle \psi^- | z_R^- \rangle, \text{ but only for } t \geq 0. \quad (44)$$

Thus the Gamow vectors are eigenvectors of the time evolution operator given by (44) and the time evolution operator on Gamow vectors, $U_+^\times(t) = e^{-iH^\times t}$, is only a semigroup. After one was led to this conclusion and has accepted the semigroup evolution for the Gamow vectors, one will have no problems to generalize to the whole space Φ_+^\times .

The time evolution operator in the space of microphysical states Φ_+^\times (and also the time evolution operator in the space Φ_-^\times) is described by a semigroup $U_+^\times(t)$ for $t \geq 0$ only (analogously in the space Φ_-^\times the time evolution is given by a semigroup $U_-^\times(t)$ for $t \leq 0$). In the RHS's (34) and (35) one has the two extensions of the Hilbert space operator $U^\dagger(t)$ ²¹

$$\text{the conjugate of } U|_{\Phi_-} : U^\dagger(t) \subset U_-^\times = e^{-iH^\times t/\hbar}; \text{ for } t \leq 0 \quad (45)$$

$$\text{the conjugate of } U|_{\Phi_+} : U^\dagger(t) \subset U_+^\times = e^{-iH^\times t/\hbar}; \text{ for } t \geq 0 \quad (46)$$

where U_\pm^\times denote the extensions of the unitary operator $U^\dagger(t)$ to the spaces Φ_\pm^\times [14]. Mathematically, U_-^\times in Φ_-^\times can be defined by (30) only for values of parameters $t \leq 0$, since for $t > 0$ the operator $U(t)|_{\Phi_-}$ is not a continuous operator which maps Φ_- into Φ_- . By the same argument, U_+^\times in Φ_+^\times can only be defined by (30) for values of parameters $t \geq 0$ because for $t < 0$, $U(t)|_{\Phi_+}$ is not a continuous operator which maps Φ_+ into Φ_+ as required by (30).

These are the mathematical arguments by which the semigroup time evolution is derived from the mathematical properties of the spaces Φ_- and Φ_+ , which in Sect. 6 had been con-

²⁰cf. Sect. XXI.4, 3rd Ed. of [24], and [37].

²¹Regarding the notation $e^{-iH^\times t}$: $U(t)$ is the unitary operator in \mathcal{H} and $e^{iHt} \equiv \sum_{n=0}^{\infty} \frac{(it)^n}{n!} H^n$ where the generator of $U(t)$, H , is defined on a dense subspace of \mathcal{H} , the space of differentiable vectors \mathcal{D} , and the infinite series converges on the dense subspace $\mathcal{A} \subset \mathcal{D} \subset \mathcal{H}$ of “analytic vectors”. The topological notions of dense, convergence, differentiable, etc., all refer to the topology in \mathcal{H} given by the norm, i.e. $U(t)$ and $U^\dagger(t)$ are \mathcal{H} -continuous (implying \mathcal{H} -bounded) operators that are \mathcal{H} -dense, \mathcal{H} -convergent, etc., in \mathcal{H} . The restriction of $U(t)$ to the subspace Φ_+ , $U_+(t) = U(t)|_{\Phi_+}$ for $t \geq 0$ only, is a Φ -continuous operator; $U_+^\times(t)$ is its conjugate as defined by (30). If one applies $U_+^\times(t)$ to a generalized eigenvector $F \in \Phi_+^\times$ of H^\times with an eigenvalue ω as defined by (31), then one can show that $U_+^\times(t)F = e^{-i\omega t}F$. For this reason we use the notation $U_+^\times(t) = e^{-iH^\times t}$. The operator H^\times is a Φ^\times -continuous operator in Φ_+^\times and one can define $\sum_{n=0}^N \frac{(it)^n}{n!} H^n \equiv U_+^N$ for all $t \geq 0$. One would usually want to denote by the exponential $e^{-iH^\times t}$ the limit with respect to the topology in Φ^\times of the sequence U_+^N for $N \rightarrow \infty$. Whether $U_+^N \rightarrow U_+^\times(t)$ for $N \rightarrow \infty$ and/or whether there is a Φ -dense subspace of vectors $F \in \Phi_+^\times$ on which $U_+^N F$ (which one would then call Φ_+^\times -analytic vectors) is not known to us.

tured from the intuitive notion of causality. However without the straightforward mathematical derivation of (44) from (43) and just on the basis of “philosophical” causality arguments alone, one would probably not have been willing to come up with these mathematical properties of Φ_- and Φ_+ that have such drastic consequences like the semigroup evolution. As one can see from the details [14, 24] of the above derivation, the reverse conclusion is also correct: The choice of the Hardy class spaces and therewith the preparation→registration arrow follows if one requires the existence of the Gamow vectors with the above properties.

The semigroup time evolution of the Gamow vectors is the mathematical expression of irreversibility for the microphysical decaying states. As a consequence of the quantum mechanical irreversibility, one can no longer calculate for every state $W^G(t) = |\psi^G(t)\rangle\langle\psi^G(t)|$, with $\psi^G \in \Phi_+^\times$, another state $W^{\text{neg}}(t) \equiv |\psi^G(-t)\rangle\langle\psi^G(-t)|$. This is empirically correct but it also leads immediately to the question whether and how this irreversibility can be compatible with the definition of a time reversal operator A_T ²², since the state obtained from a given state $W(t)$ by A_T transformation,

$$W^T \equiv A_T^{-1}W(t)A_T, \quad (47)$$

is usually identified with the state $W^{\text{neg}} \equiv W(-t)$, i.e. is assumed to have the property

$$\begin{aligned} A_T^{-1}W(t)A_T &= W(-t) \\ (\text{or } \phi^T(\mathbf{x}, t) &= \phi(\mathbf{x}, -t) = \phi^*(\mathbf{x}, t) \text{ for wave functions}). \end{aligned} \quad (48)$$

The answer to this question is that neither the time reversed state $W^T(t)$ nor the backward time translated state $W^{\text{neg}}(t)$ is in general physically defined. For example, in a typical scattering experiment the “out-states” represent highly correlated spherical waves whereas the prepared in-states are typically two uncorrelated plane waves (e.g., two colliding monochromatic beams). The time reversal of this experiment would require a preparation apparatus that prepares highly correlated (with fixed phase relationship) incoming spherical waves that would be scattered into two uncorrelated plane waves. An experimental setup that would accomplish this would have to be so complicated that it is impossible to build, at least in this world. Thus, not for every preparable W can one prepare a state which would be described by its time reversal transformed $W^T = A_T^{-1}WA_T$ (for another example see [36]).

The time reversal operator A_T is not defined by its action on states, but by its relation to the observables. Examples of these relations are

$$\begin{aligned} A_TP_iA_T^{-1} &= -P_i, & A_TJ_iA_T^{-1} &= -J_i, \\ A_TU_PA_T^{-1} &= \varepsilon_T\varepsilon_IU_P, & A_THA_T^{-1} &= H, \\ A_TH_0A_T^{-1} &= H_0, & A_TSA_T^{-1} &= S^\dagger = S^{-1}, \end{aligned} \quad (49)$$

²²In place of A_T one could as well use the CPT-operator here with minor modifications.

and they follow from the extended projective representations of the extended Poincaré group [46]. The generators P_i, H, J_i represent momentum, energy, angular momentum, respectively, the S -operator is a complicated function of the interaction Hamiltonian $V = H - H_0$ and U_P is the unitary and hermitian parity operator normalized to $U_P^2 = 1$. The quantities

$$\varepsilon_T = A_T^2, \text{ and } \varepsilon_I = (U_P A_T)^2 \equiv A_I^2 \quad (50)$$

are real phase factors which define the four different extensions of the restricted space-time symmetry transformations by space inversion $P = g$, time inversion $T = -g$ and the space-time inversion $I = PT = -1$, which were derived by Wigner [47]. Of the four possible extensions characterized by the four pairs of phase factor $(\varepsilon_T, \varepsilon_I) = (\pm 1, \pm 1)$ the only extensions used in relativistic field theory [46] are those characterized by

$$\varepsilon_T = (-1)^{2j} \varepsilon_I = (-1)^{2j} \text{ where } j \text{ is the spin.} \quad (51)$$

With this choice the time reversal operator A_T has the following transformation property:

$$A_T : \Phi_{\mp} \rightarrow \Phi_{\pm}; \quad \Phi_+ \ni \psi^- = A_T \phi^+, \phi^+ \in \Phi_-. \quad (52)$$

This is also the solution suggested by conventional scattering theory where the *in*-states ϕ^+ are the time reverse of the so-called *out*-states ψ^- . The “out-states” ψ^- are actually observables and not states because they are specified by the detector whereas in-states ϕ^+ are specified by the preparation apparatus (accelerator). In our interpretation, the transformation property (52) means that states are transformed into observables and vice-versa. This requires the identification of the set of states with the set of observables (i.e., no arrow of time) and to assign to every $W(t)$ a $W^T(t) \equiv A_T^{-1} W(t) A_T$ fulfilling (48). This means that in the case (51) one cannot have irreversibility in the sense described above, which as mentioned in Sect. 3 is in contradiction to at least some arguments concerning the improbability to prepare time reversed states (cf. remark above and Chap. 13.2 in [36]).

Fortunately, there are three other classes of representations derived by Wigner [47] and one can choose instead of (51) one of the non-standard extensions for A_T which lead to a doubling of the spaces (time reversal doubling [47]). We suggest the choice:

$$\varepsilon_T = -(-1)^{2j}, \quad \varepsilon_I = -(-1)^{2j}. \quad (53)$$

The A_T which fulfills (53) can be shown to be compatible with the microphysical irreversibility of (45) and (46) [48].

7.2 Gamow Vectors

“The data suggest that a particle decays if it can and that it is stable only if there is no state . . . to which it is allowed to decay. Stability does not appear to be a criterion for *elementarity*.”

—Frauenfelder and Henly²³

The RHS formulation accounts for the Dirac kets; they are defined as generalized eigenvectors over the spaces Φ_+ , Φ_- , and $\Phi = \Phi_+ + \Phi_-$ with generalized eigenvalues E of the Hamiltonian H :

$$\langle H\phi|E^\pm\rangle = \langle\phi|H^\times|E^\pm\rangle = E\langle\phi|E^\pm\rangle \text{ for all } \phi \in \Phi, \quad (54)$$

where the E represent the scattering energies (continuous spectrum of \bar{H}). This is no surprise because the RHS formulation was devised to provide a mathematical justification for the Dirac scattering states.

Unforeseen was that the RHS formulation also accounts for the Gamow kets. The decaying Gamow kets $\psi^G = |E_R - i\Gamma/2^-\rangle\sqrt{2\pi\Gamma} = |z_R^-\rangle\sqrt{2\pi\Gamma}$ are defined as generalized eigenvectors over the space Φ_+ with generalized eigenvalue $(E_R - i\Gamma/2)$:

$$\langle H\psi^-|E_R - i\Gamma/2^-\rangle \equiv \langle\psi^-|H^\times|E_R - i\Gamma/2^-\rangle = (E_R - i\Gamma/2)\langle\psi^-|E_R - i\Gamma/2^-\rangle, \quad (55a)$$

or its complex conjugate:

$$\langle\psi^G|H|\psi^-\rangle = \langle\psi^G|\psi^-\rangle(E_R + i\Gamma/2), \text{ for all } \psi^- \in \Phi_+. \quad (55b)$$

These Gamow vectors $\psi^G = |E_R - i\Gamma/2^-\rangle\sqrt{2\pi\Gamma}$ have the following properties:

- (1) They are derived as functionals of the resonance pole term at $z_R = E_R - i\Gamma/2$ in the lower half of the second sheet of the analytically continued S -matrix.
- (2) They have Breit-Wigner energy distribution

$$\langle^-E|\psi^G\rangle = i\sqrt{\Gamma/2\pi}\frac{1}{E - (E_R - i\Gamma/2)}, \quad -\infty_{II} < E < \infty, \quad (56)$$

where E is on the second sheet for negative values²⁴.

²³p. 91-92 of [30].

²⁴ The “wave function” $\langle^-E|z_R^-\rangle$ of the generalized vector $|z_R^-\rangle$ in (56) is represented by a regular function (even of Hardy class \mathcal{H}_+^2) though one would expect it to be a non-regular distribution. The latter is indeed correct because (56) is not the actual wave function. The wave function is a function on the continuous spectrum $\{0 \leq E \leq \infty\}$ and over $\mathcal{S} \cap \mathcal{H}_+^2|_{\mathbb{R}^+}$, i.e. as an element of $(\mathcal{S} \cap \mathcal{H}_+^2|_{\mathbb{R}^+})^\times$, $\langle^-E|z_R^-\rangle$ is a distribution—not a function (regular distribution). However in (56) we need the whole real axis $\{-\infty < E < \infty\}$ to represent it as a regular function. Extending E in (56) to the negative real axis does however not mean that we have let the spectrum of H go to $-\infty$, because the negative values of E in (56) are the result of an analytic continuation into the second sheet. For more detail see p. 504-505 of [14], or see [49], where an explicit expression for the distribution $\langle^-E|z_R^-\rangle \in (\mathcal{S} \cap \mathcal{H}_+^2|_{\mathbb{R}^+})^\times$ has been given.

- (3) The time evolution of the Gamow vectors is derived from the pole term of the S -matrix as:

$$\langle e^{iHt}\psi^-|z_R^- \rangle \equiv \langle \psi^-|e^{-iH^\times t}|z_R^- \rangle = e^{-iE_R t} e^{-(\Gamma/2)t} \langle \psi^-|z_R^- \rangle \quad (57a)$$

or for the complex conjugate:

$$\langle^- z_R|e^{iHt}|\psi^- \rangle = e^{iE_R t} e^{-(\Gamma/2)t} \langle^- z_R|\psi^- \rangle, \text{ for every } \psi^- \in \Phi_+ \text{ and for } t \geq 0. \quad (57b)$$

The above properties—except for the semigroup property—are historically ascribed to the empirical notion of decaying states and resonances. Equations (55) and (57) give the precise mathematical form as a distribution. Therefore the Gamow kets justly deserve the name resonance state vectors with complex resonance energy $z_R = E_R - i\Gamma/2$. The property (2) identifies E_R as the resonance energy and Γ as the resonance width. The property (3) shows that $\tau_R = 1/\Gamma (= \hbar/\Gamma)$ is the lifetime of the decaying resonance state.

The semigroup evolution (57) expresses the time-asymmetry of the Gamow vector and the resonance state which it describes. It is not a condition that we demanded of the $|z_R^- \rangle$, like we demanded (40) or some other mathematical formulation of causality, but it can be derived from the Hardy class property (35) which is some mathematical statement of causality²⁵.

In the same way as stationary states are given by bound state poles of the S -matrix and described as eigenvectors of H with real (negative) eigenvalues, decaying states are given by resonance poles and described as generalized eigenvectors of H with complex eigenvalue. This puts stable and decaying particles on the same footing and shows that resonances can be thought of as autonomous quantum physical entities which are not less fundamental than stable particles.

Instead of defining the Gamow vectors from the poles of the analytically continued S -matrix, the Gamow vectors can also be defined from the poles of the extended resolvent of the Hamiltonian [50]. These definitions are equivalent in many cases in which the S -matrix is obtained from the Hamiltonian. The S -matrix definition can be used in more general settings, e.g. in the relativistic case when one can start from the unitary representations of the Poincaré group in \mathcal{H} and then extend it to Φ_\pm^\times .

One can generalize the Gamow vectors to Gamow-Jordan vectors associated to higher-order poles of the S -matrix [51] and obtain higher-order Gamow states [52]. These vectors

²⁵It is remarkable that the asymmetry of the time evolution of the resonance states (57)—or more generally of the semigroup (46)—is the same as the preparation→registration arrow of time (40). This is established mathematically by the use of the Hardy class RHS's (34) and (35). Since equation (40) applied to the quantum theory of cosmology is, according to (7), the time-asymmetry of the universe [8], the mathematical idealization given by the RHS formulation of quantum mechanics would establish a connection between the time-asymmetry of quantum cosmology and the irreversibility of quantum decay phenomena.

are also functionals over Hardy class spaces and have semigroup time evolutions. They have some features which are heuristically associated with higher-order poles of the S -matrix [53] like polynomial time dependence. But non-reducible Gamow states that follow from the higher-order S -matrix pole term have purely exponential time evolution, which is a feature that was not expected. Since so far there is no empirical evidence for states associated with higher-order poles of the S -matrix, we shall not discuss them here any further.

7.3 Generalized Basis Vector Expansions

The most important consequence of the RHS formulation are the generalized basis vector expansions. The expansion (14) that Dirac envisioned has been proven as the nuclear spectral theorem. This theorem requires the nuclearity of the space Φ (or a little less [18]), a mathematical subtlety unimportant for physics. What is important for physics is that (14) holds in the space Φ ; therefore we will call a triplet of spaces (27) a RHS only if the necessary conditions for the nuclear spectral theorem are fulfilled.

All basis vector expansions are generalizations of the elementary basis vector expansion of a vector in \mathbb{R}^3 ,

$$\mathbf{x} = \sum_{i=1}^3 \hat{\mathbf{e}}_i (\hat{\mathbf{e}}_i \cdot \mathbf{x}) = \sum_{i=1}^3 \hat{\mathbf{e}}_i \cdot x_i, \quad (58)$$

where $\hat{\mathbf{e}}_i$ are chosen to be the physically distinguished basis vectors.

Earlier generalizations of this are the fundamental theorem of linear algebra which states that for every self-adjoint operator H in an n -dimensional Euclidean space \mathcal{H}_n there exists a complete basis system $e_1 \dots e_n$ in \mathcal{H}_n of eigenvectors:

$$\begin{aligned} H e_i &= E_i e_i, \quad e_i \in \mathcal{H}_n \quad (i = 1, 2, \dots, n), \text{ such that} \\ f &= \sum_{i=1}^n e_i (e_i | f) \quad \text{for every } f \in \mathcal{H}_n \end{aligned} \quad (59)$$

This theorem generalizes to the infinite dimensional Hilbert space \mathcal{H} , but only for self-adjoint operators H which are completely continuous²⁶. For an arbitrary self-adjoint operator H one cannot find a complete system of eigenvectors in \mathcal{H} (complete in the sense that every $f \in \mathcal{H}$ can be expanded in the form (59)). Many physically important operators do not have even a single eigenvectors in \mathcal{H} . Because $|(e_i, f)|^2$ represents the probability to measure the value E_i of the observable H in the state f , one wants to use a basis system of eigenvectors for the

²⁶These are also called compact operators and include Hilbert-Schmidt, nuclear and trace-class operators.

distinguished observables not an arbitrary basis system, which one can always find by the Schmidt orthonormalization procedure. In order to obtain such a basis system of eigenvectors for an arbitrary observable, one has to go outside the HS; they are generalized eigenvectors as defined by (31).

The *first* step outside the HS is to use the nuclear spectral theorem to justify the Dirac basis vector expansion (14) in terms of well-defined mathematical quantities. The nuclear spectral theorem states that for every $\phi \in \Phi$ (not for every $f \in \mathcal{H}$) one can find a complete set of eigenvectors in Φ^\times (not in \mathcal{H}):

$$\phi = \int_0^{+\infty} dE |E^+\rangle \langle^+ E | \phi\rangle + \sum_n |E_n\rangle (E_n | \phi) \text{ for every } \phi \in \Phi. \quad (60a)$$

Since $\phi^+ \in \Phi_- \subset \Phi$, of course the same expansion holds for ϕ^+ :

$$\phi^+ = \int_0^{+\infty} dE |E^+\rangle \langle^+ E | \phi^+\rangle + \sum_n |E_n\rangle (E_n | \phi^+). \quad (60b)$$

In the above expansion, $|E_n\rangle$ are the discrete eigenvectors of the exact Hamiltonian $H = K + V$, $H|E_n\rangle = E|E_n\rangle$, which describe bound states. The generalized eigenvectors (Dirac kets) of H , $|E^+\rangle \in \Phi^\times$, fulfill $\langle H\chi | E^+\rangle = \langle \chi | H^\times | E^+\rangle = E \langle \chi | E^+\rangle$ for all $\chi \in \Phi$, cf. (31). The “coordinates” of the vector ϕ with respect to the continuous basis $|E^+\rangle$, i.e., the set of energy wave functions $\langle^+ E | \phi^+\rangle$, form a realization of the space Φ by a space of functions (in the same way as the coordinates x_i form a “realization” of the vector \mathbf{x}). We call the vector $\phi \in \Phi$ “well-behaved” if $\langle^+ E | \phi\rangle$ is a well-behaved function, i.e., $\langle^+ E | \phi\rangle \in \mathcal{S}$. The $|E^+\rangle$ correspond to the continuous spectrum of H (the “scattering states”) and the integration extends over the continuous spectrum ²⁷ $0 \leq E < \infty$ (the physical scattering energies).

²⁷ The nuclear spectral theorem actually asserts much less than (60) but it applies to a much larger set of operators A than the operators needed in quantum physics:

Let A be *any* self-adjoint (or unitary or normal) operator in the Hilbert space \mathcal{H} (the analogous statement holds for a strongly commuting family of operators A_1, A_2, \dots, A_N) of the rigged Hilbert space $\Phi \subset \mathcal{H} \subset \Phi^\times$, and Λ its Hilbert space spectrum. Then there exists a unique measure μ on Λ such that for $\phi, \psi \in \Phi$

$$(\psi, \phi) = \int_\Lambda d\mu(\lambda) \langle \psi | \lambda \rangle \lambda \phi \quad (60')$$

where the $|\lambda\rangle \in \Phi^\times$ are the generalized eigenvectors of A , i.e.

$$\langle A\phi | \lambda \rangle \equiv \langle \phi | A^\times | \lambda \rangle = \lambda \langle \phi | \lambda \rangle, \text{ for } \mu\text{-almost every } \lambda. \quad (60'')$$

The measure $d\mu(\lambda)$ depends upon the operator A , and for a general operator μ consists of three parts:

1. the discrete spectrum, $d\mu(\lambda) = \sum_n \delta(\lambda - \lambda_n) d\lambda$,

If the *out*-wave functions are more readily available, one would have chosen the $|E^- \rangle$, defined as $|E^+ \rangle = |E^- \rangle S(E + i0)$, in place of the $|E^+ \rangle$ in (60), or one could have chosen any other vector which differs from $|E^+ \rangle$ by an energy-dependent phase factor²⁸.

$$\begin{aligned}\phi &= \int_0^{+\infty} dE |E^- \rangle S(E + i0) \langle^+ E | \phi \rangle + \sum_n |E_n \rangle (E_n | \phi) \\ \phi &= \int_0^{+\infty} dE |E^- \rangle \langle^- E | \phi \rangle + \sum_n |E_n \rangle (E_n | \phi) \quad \text{for every } \phi \in \Phi\end{aligned}\tag{61}$$

In a scattering experiment the following S -matrix elements are measured

$$\begin{aligned}(\psi^{\text{out}}, \phi^{\text{out}}) &= (\psi^{\text{out}}, S\phi^{\text{in}}) = (\psi^-, \phi^+) \\ &= \int_0^{+\infty} \int_0^{+\infty} dE dE' \langle \psi^{\text{out}} | E \rangle \langle E | S | E' \rangle \langle E' | \phi^{\text{in}} \rangle \\ &= \int_0^{+\infty} dE \langle^- \psi | E^- \rangle S(E + i0) \langle^+ E | \phi^+ \rangle\end{aligned}\tag{62}$$

-
- 2. the absolutely continuous spectrum, $d\mu(\lambda) = \rho(\lambda)d\lambda$, and
 - 3. the singularly continuous spectrum.

The discrete spectrum corresponds to the sum in (60), the absolutely continuous spectrum corresponds to the integral in (60) and the singularly continuous spectrum has never occurred for operators that represent physical observables.

The observables in quantum mechanics are usually derived from representations of space-time symmetry groups, spectrum generating groups, intrinsic symmetry groups or other Lie groups. Therefore in physics the measure $d\mu$ is the Plancherel measure of compact or non-compact Lie groups and their compact or non-compact subgroups. For the compact case, one has only case (1); for the non-compact Lie groups that have occurred in physics (e.g. all Abelian groups or all classical groups, semi-direct product and more) one has case (2) with $\rho(\lambda)$ being a continuous function. In these cases one has the “normalizations” of the generalized eigenvectors:

$$\langle \lambda_n | \lambda_{n'} \rangle = \delta_{nn'} \quad \langle \lambda | \lambda' \rangle = \rho^{-1}(\lambda) \delta(\lambda - \lambda').$$

After redefinition of the kets, $|\lambda \rangle \rightarrow |E^+ \rangle = |\lambda \rangle \rho^{\frac{1}{2}}(\lambda)$ and $\lambda \rightarrow E$, this gives (60a) with $\langle E | E' \rangle = \delta(E - E')$. The integral in (60') is still a Lebesgue integral and (60') holds for $\phi, \psi \in \mathcal{H}$ if $\langle \lambda | \phi \rangle, \langle \lambda | \psi \rangle \in \mathcal{L}^2(\lambda)$. However now we can follow the procedure described in Sect. 3 and use only the smooth functions $\langle \lambda | \phi \rangle, \langle \lambda | \psi \rangle \in \mathcal{S}_\Lambda$ under the integral in (60'). Then for the subspace Φ whose realization is given by the smooth functions \mathcal{S} we can use in (60') Riemann integration with (60'') holding for every λ (rather than μ -almost every λ). No example of a physical observable is known in which the integrals (60) can not be interpreted as Riemann integrals and the eigenvalue equation is valid only for μ -almost all λ . We will always presume that the observables have discrete and/or absolutely continuous spectra. Dirac's visionary tools of quantum mechanics, (13) and (14), have thus been made rigorous.

²⁸ If the additional quantum numbers (e.g., channel labels) η would be considered, then in place of the phase factor $S(E) = e^{2i\delta(E)}$ one would of course have a unitary matrix, $|E, \eta^+ \rangle = |E, \eta'^- \rangle \langle \eta' | S(E) | \eta \rangle$.

In here ψ^- and ϕ^+ are very well-behaved vectors with:

$\phi^+ \in \Phi_-$ representing the state prepared by the accelerator, cf. (34) and
 $\psi^- \in \Phi_+$ representing the observable registered by the detector, cf. (35).

This means that the functions $\langle E|\psi^{\text{out}}\rangle = \langle^-E|\psi^-\rangle$ are “very well-behaved functions from above” and $\langle E|\phi^{\text{in}}\rangle = \langle^+E|\phi^+\rangle$ are “very well-behaved functions from below”, i.e., equations (36) and (37):

$$\langle E|\phi^{\text{in}}\rangle = \langle^+E|\phi^+\rangle \in \mathcal{S} \cap \mathcal{H}_-^2|_{\mathbb{R}^+} \quad (63a)$$

$$\langle E|\psi^{\text{out}}\rangle = \langle^-E|\psi^-\rangle \in \mathcal{S} \cap \mathcal{H}_+^2|_{\mathbb{R}^+}. \quad (63b)$$

In the scattering experiment, the wave functions $\langle^+E|\phi^+\rangle$ (the components of the vector ϕ^+ along $|E^+\rangle$) represent the probability $\int_{\Delta E} dE |\langle^+E|\phi^+\rangle|^2$ that the beam prepared by the accelerator has an energy in the interval ΔE , i.e., $|\langle^+E|\phi^+\rangle|^2$ is the energy distribution in the beam state ϕ^+ and $|\langle^-E|\psi^-\rangle|^2$ is the energy resolution of the detector ψ^- (the detector efficiency). But the components of ϕ^+ along $|E^-\rangle$, and of ψ^- along $|E^+\rangle$, do not just represent apparatus properties, but the properties of the scattering system.

If we consider for the vector ϕ in (61) a very well-behaved vector $\phi^+ \in \Phi_- \subset \Phi$

$$\phi^+ = \int_0^\infty dE |E^-\rangle \langle^-E|\phi^+\rangle + \sum |E_n\rangle (E_n|\phi^+\rangle) \quad (64)$$

then we obtain a Dirac basis vector expansion of ϕ^+ with respect to the generalized eigenvectors $|E^-\rangle$ of H . Although both expansions (60b) and (64) use generalized eigenvectors of H , the basis system used in expansion (64) differs from that of (60b) by a phase factor²⁹. Whereas the components $\langle^+E|\phi^+\rangle$ along $|E^+\rangle$ contain only information about the preparation apparatus, the components $\langle^-E|\phi^+\rangle$ are the interaction wave functions describing also the dynamics (analogous considerations hold for “out-state” (observable) vectors $\psi^- \in \Phi_+ \subset \Phi$).

The *second* step on the way outside the HS is the *complex basis vector expansion*. It holds for “very well-behaved” vectors, i.e., for vectors of a subspace $\Phi_- \subset \Phi$ only. For every $\phi^+ \in \Phi_-$ (and a similar expression holds also for every $\psi^- \in \Phi_+$), one obtains the following basis vector expansion for the case of an S -matrix with a finite number of resonance poles at the positions $z_{R_i} = E_{R_i} - i\Gamma_i/2$, $i = 1, 2, \dots, N$:

$$\phi^+ = \int_0^{-\infty II} dE |E^-\rangle \langle^-E|\phi^+\rangle - \sum_{i=1}^N |z_{R_i}^-\rangle 2\pi\Gamma_i \langle^+z_{R_i}|\phi^+\rangle + \sum_n |E_n\rangle (E_n|\phi^+\rangle)$$

²⁹cf. footnote (28).

$$\text{for } \phi^+ \in \Phi_- . \quad (65)$$

Here $|z_{R_i}^- \rangle \sqrt{2\pi\Gamma_i} = \psi^{G_i} \in \Phi_+^\times$ are Gamow kets (55) representing decaying states (57a)³⁰. The remarkable feature of the basis vector expansion (65) is that the decaying states $|z_{R_i}^- \rangle$ appear on the same footing as the stationary states $|E_n \rangle$ ³¹.

The last term in the expansions (60), (64) and (65) will be absent if there are no bound states; we shall omit this term in the following discussions. The first term in (65) is the background integral related with the background phase shifts. The integration, taken along the negative real axis in the second sheet (for which the values of $\langle^- E|\phi^+ \rangle = S(E)\langle^+ E|\phi^+ \rangle$ can be calculated from the experimental values $\langle E|\phi^{\text{in}} \rangle$ for positive energies using the van Winter theorem [54]), could be deformed into integration over many other equivalent contours in the lower half plane of the second sheet, if those would be more convenient to calculate.

³⁰There is a corresponding basis vector expansion of $\psi^- \in \Phi_+$; in place of $|z_{R_i}^- \rangle$, it contains generalized eigenvectors $|z_{R_i}^{*+} \rangle$ with eigenvalue $z_{R_i}^* = E_{R_i} + i\frac{\Gamma_i}{2}$. These are the Gamow vectors associated with the S -matrix pole at $z_{R_i}^*$. They have an exponentially growing semigroup evolution for $-\infty < t \leq 0$.

³¹The forms (60) and (65) of the generalized basis vector expansions assume that H is the only observable to be diagonalized (cyclic operator). If the complete system of commuting observables (c.s.c.o.) consists of $(N+1)$ operators $H, B_{(1)}, B_{(2)}, \dots, B_{(N)} \equiv H, B$, then we have to make the following replacements for the projection operators

$$|E_n \rangle \langle E_n| \rightarrow \sum_b |E_n, b \rangle \langle E_n, b|,$$

where the sum extends over all values of the degeneracy quantum numbers $b = b_{(1)}, b_{(2)} \dots b_{(N)}$ of the energy E_n . Similarly, in (60) to (65) we have to make the replacements:

$$\begin{aligned} |E^+ \rangle \langle^+ E| &\rightarrow \sum_b |E, b^+ \rangle \langle^+ E, b| \\ &= \int d\mu(b_1, b_2, \dots) \sum_{b_d, \dots, b_N} |E, b_1 \dots b_N^+ \rangle \langle^+ E, b_1 \dots b_N| \\ |z_R^- \rangle \langle^+ z_R| &\rightarrow \sum_b |z_R, b^- \rangle \langle^+ z_R, b| \\ &= \int d\mu(b_1, b_2, \dots) \sum_{b_d, \dots, b_N} |z_R, b_1 \dots b_N^- \rangle \langle^+ z_R, b_1 \dots b_N| \end{aligned} \quad (66)$$

where b_1, b_2, \dots are the continuous and b_d, \dots, b_N are the discrete degeneracy quantum numbers. The operators B could be, e.g., the orbital angular momentum operators J_3, \mathbf{J}^2 if we have a spherically symmetric (spin-less) scattering system $[H, J_i] = 0$; then the quantum numbers $b = b_1, b_2$ are $b = j, j_3$. They could be the momentum operators P_i if $[H, P_1] = 0$; then $(E, b_1, b_2) = (p_1, p_2, p_3)$ or $(E, b_1, b_2) = (E, \theta_p, \varphi_p)$, where (θ_p, φ_p) are the spherical coordinates of \mathbf{p} and $d\mu(b_1, b_2) = d\cos\theta_p d\varphi_p$. The labels b could also be some intrinsic quantum numbers, like charges or the channel label η .

For a $\phi^+ \in \Phi_- \subset \Phi$, both expansions (64) and (65), which use two different basis systems, hold. The expansion (65) separates the individual resonance poles, whereas (64) has the resonances contained together with the “background” in the energy wave function $\langle^-E|\phi^+\rangle$. The function $|\langle^-E|\phi^+\rangle|^2$ may have a bump at $E = E_R$, but $\langle^-E|\phi^+\rangle \in \mathcal{S}$ cannot be the Breit-Wigner distribution (56) characteristic of a resonance because the Breit-Wigner distribution is not a well-behaved function³². The complex energy basis vector expansion (65) is the much preferred representation for investigating resonances.

For the sake of definiteness we shall now assume that there are two decaying states $R_1 = S$ and $R_2 = L$ and no bound states. According to the expansion (65), the pure state (prepared by the experimental apparatus) has the following representation in terms of the Gamow vectors $\psi_L^G = -|z_L^-\rangle\sqrt{2\pi\Gamma_L}$, $\psi_S^G = -|z_S^-\rangle\sqrt{2\pi\Gamma_S}$ and the remaining part which we call ϕ_{bg}^+ (the background):

$$\phi^+ = \psi_L^G b_L + \psi_S^G b_S + \int_0^{-\infty_{II}} dE |E^+\rangle \langle^+E|\phi^+\rangle. \quad (67)$$

In here b_L and b_S are some complex numbers that depend on the normalization of the Gamow vectors ψ_L^G , ψ_S^G (and of ϕ^+), and upon some phase convention. All the vectors in the generalized basis system expansion are (generalized) eigenvectors of the exact Hamiltonian, and, in particular, the Gamow vectors ψ_L^G , ψ_S^G are eigenvectors of the exact Hamiltonian with complex eigenvalue $(E_L - i\Gamma_L/2)$ and $(E_S - i\Gamma_S/2)$, respectively. If we ignore $\phi_{\text{bg}}^+ = \int_0^{-\infty_{II}} dE |E^+\rangle \langle^+E|\phi^+\rangle$ then ϕ^+ in (67) is the superposition of two eigenvectors of H with complex eigenvalues z_L and z_S ; the Hamiltonian matrix is complex and diagonalizable.

We now apply the time evolution operator to (67). Since the ψ_L^G , ψ_S^G are elements of Φ_+^\times we apply the operator $U_+^\times(t)$ of (45) and obtain:

$$\phi^+(t) \equiv e^{-iH^\times t} \phi^+ = e^{-i(E_L - i\Gamma_L/2)t} \psi_L^G b_L + e^{-i(E_S - i\Gamma_S/2)t} \psi_S^G b_S + \phi_{\text{bg}}^+(t); \quad t \geq 0. \quad (68)$$

Since the time evolution semigroup (45) has the restriction $t \geq 0$, the same restriction must be used for (68). The time evolved background term is

$$\phi_{\text{bg}}^+(t) \equiv \int_0^{-\infty_{II}} dE e^{iEt} |E^-\rangle S(E) \langle^+E|\phi^+\rangle; \quad t \geq 0. \quad (69)$$

These equations ((65), (67) through (69)) are understood as a functional equation over all $\psi^- \in \Phi_+$. This means that these expansions of $\phi^+(t) \in \Phi_- \subset \Phi_+^\times$ can be used to

³²Note that the Gamow ket has an “idealized” Breit-Wigner energy distribution $\langle^-E|\psi^G\rangle$ (56) which extends from $-\infty_{II}$ to $+\infty_{II} = \infty_I$. This is a regular function on \mathbb{R} , but ψ^G cannot be realized by a regular function on the physical energy values $\mathbb{R}_+ = \{E | 0 \leq E < \infty_I\}$ since ψ^G is a generalized vector and correspondingly $\langle^-E|\psi^G\rangle$ is a distribution over $\mathcal{S} \cap \mathcal{H}_+^2|_{\mathbb{R}_+}$, cf. also footnote 24.

obtain $\langle \psi^- | \phi^+(t) \rangle$ (whose modulus square is the probability to find the time evolved state by a detector that detects the observable $|\psi^- \rangle \langle \psi^-|$) for any $\psi^- \in \Phi_+$, but *not* to calculate $\langle \psi | \phi \rangle$ for a $\psi \in \Phi_-$. This is not a problem because the expressions $|\langle \psi | \phi^+(t) \rangle|^2$, $\psi \in \Phi_-$ and $\phi^+ \in \Phi_-$, have no physical meaning, since they would represent the probability for finding an in-state ψ in an in-state $\phi^+(t)$ at $t \geq 0$ which is not measurable in a scattering experiment. The semigroup time evolution operator $U_+^\times(t)$ can also be applied to the basis vector expansion (64) with ϕ^+ understood as functional $\phi^+ \in \Phi_-^\times$ over the $\psi^- \in \Phi_+$ because (64) is a functional equation over Φ_+ since the $\langle -E | \psi^- \rangle$ are according to (37) elements of $\mathcal{S} \cap \mathcal{H}_+^2|_{\mathbb{R}^+}$. However the semigroup operator $U_+^\times(t)$ cannot be applied to the representation (60b) for $\phi^+ \in \Phi_- \subset \Phi_+^\times$, because it is *not* a functional equation over Φ_+ since the $\langle +E | \psi^- \rangle$, with $\psi^- \in \Phi_+$, are not necessarily elements of $\mathcal{S} \cap \mathcal{H}_+^2|_{\mathbb{R}^+}$.

Of the two exact but different basis vector expansions (60b) and (65) for the same $\phi^+ \in \Phi_-$, (60b) is the standard expansion and has a correspondence in the Hilbert space (spectral resolution of operators with a continuous spectrum). The expansion (65) is new and shows that the quasi-stationary states $|z_{R_i}^- \rangle$ can serve as basis vectors in very much the same manner as the stationary states $|E_n \rangle$ in the standard case. But in addition to the resonance states the new basis vector expansion (65) (for any $N \neq 0$, e.g. $N = 1$ or $N = 2$ as in (68)) also contains an integral over the negative real axis from $E = -\infty_{II}$ to 0 in the second sheet of the energy surface of the S -matrix. This integral depends on the preparation of the state and may be infinitesimally small, but cannot be zero. Its time dependence is non-exponential and could cause observable deviations from the exponential law for the transition rate of the prepared state ϕ^+ . It may also have some other small but observable consequences.

The result (68) means that the semigroup time evolution of a superposition of two (or more) Gamow states does not regenerate one Gamow state from the background $\phi_{\text{bg}}^+(t)$ or from the other Gamow vector. In particular, if the state ϕ^+ can be prepared such that at some time $t_0 \geq 0$ the background term $\phi_{\text{bg}}^+(t)$ is practically zero, then it will remain practically zero for all $t > t_0$, and the two Gamow states will evolve separately according to separate exponential laws without regenerating each other:

$$\phi^+(t) \approx e^{-i(E_L - i\Gamma_L/2)t} \psi_L^G b_L + e^{-i(E_S - i\Gamma_S/2)t} \psi_S^G b_S; \quad t \geq 0. \quad (70)$$

Approximations like (70) have been used for the time evolution of one- and two-resonance systems (like the K_L - K_S system with $\phi^+(t)$ representing the K^0 state) in theories with “effective Hamiltonians” given by a 2×2 complex diagonalizable matrix.

The full expansion (65) (again neglecting bound states) leads to a matrix representation of the self-adjoint semi-bounded Hamiltonian H in the following form:

$$\begin{aligned}
\begin{pmatrix} \langle H\psi^- | z_{R_1}^- \rangle \\ \langle H\psi^- | z_{R_2}^- \rangle \\ \vdots \\ \langle H\psi^- | z_{R_N}^- \rangle \\ \langle H\psi^- | E^- \rangle \end{pmatrix} &= \begin{pmatrix} \langle \psi^- | H^\times | z_{R_1}^- \rangle \\ \langle \psi^- | H^\times | z_{R_2}^- \rangle \\ \vdots \\ \langle \psi^- | H^\times | z_{R_N}^- \rangle \\ \langle \psi^- | H^\times | E^- \rangle \end{pmatrix} \\
&= \begin{pmatrix} z_{R_1} & & & 0 \\ & z_{R_2} & & \vdots \\ & & \ddots & \vdots \\ & & & z_{R_N} & 0 \\ 0 & \dots & \dots & 0 & (E) \end{pmatrix} \begin{pmatrix} \langle \psi^- | z_{R_1}^- \rangle \\ \langle \psi^- | z_{R_2}^- \rangle \\ \vdots \\ \langle \psi^- | z_{R_N}^- \rangle \\ \langle \psi^- | E^- \rangle \end{pmatrix}
\end{aligned}$$

$$\psi^- \in \Phi_+ \subset \Phi, \quad -\infty_{II} < E \leq 0, \quad (71)$$

where the lowest row represents the diagonal continuously infinite real energy matrix:

$$(\langle H\psi^- | E^- \rangle) = (\langle \psi^- | H^\times | E^- \rangle) = (E) (\langle \psi^- | E^- \rangle); \quad \psi^- \in \Phi, \quad -\infty_{II} < E \leq 0. \quad (72)$$

Since the basis vector expansion (65) is an exact representation of $\phi^+ \in \Phi_-$, the matrix representation (71) also is an exact representation of the self-adjoint Hamiltonian. In the phenomenological descriptions by complex effective Hamiltonians, one uses a truncation of (71) which corresponds to omitting the background integral, i.e., omitting the whole continuously infinite diagonal matrix (E) (and sometimes even some of the z_{R_i}).

The extra term ϕ_{bg}^+ in (67) and (68) is not taken into consideration in any of the finite dimensional effective theories of complex Hamiltonians, and in particular not in the Lee-Oehme-Yang theory of the neutral Kaon system. This term, which comes from the integral along the negative real axis in the second sheet of the S -matrix, can be shown to be also decaying, i.e., $|\langle \psi^- | \phi_{\text{bg}}^+(t) \rangle|^2 \rightarrow 0$ for $t \rightarrow \infty$ for every $\psi^- \in \Phi_+$, but it decays more slowly than the exponential. The standard effective theories, like the enormously successful Lee-Oehme-Yang theory of the neutral K-system, emerge as subtheories of the exact complex basis vector expansion (65) or (67) in the N -dimensional space $\mathcal{M}_N = \left\{ \phi | \phi = \sum_{i=1}^N |z_{R_i}^- \rangle c_i \right\}$ spanned by the Gamow vectors $|z_{R_i}^- \rangle$. However \mathcal{M}_N is a subspace of Φ_+^\times and lies *outside* the standard HS. These effective theories are usually legitimized by the Wigner-Weisskopf approximation. In our irreversible quantum theory the expression (68) is exact and can be used to justify the effective theory (70) as a subtheory in $\mathcal{M}_N \subset \Phi_+^\times$ which remains invariant under time translations in the forward direction, (68). The expression (68) shows that the “deviation from the exponential decay law” does not arise for the resonance state but are the properties of the background terms (background phase shifts).

The emergence of such an enormously successful phenomenological description as the Lee-Oehme-Yang theory as a subtheory is an empirical validation of the complex basis vector expansion (65).

7.4 The Golden Rule from Fundamental Probabilities

“[T]he physical system leaves the [initial decaying] state $|\phi_i\rangle$ irreversibly.”

—Cohen-Tannoudji, et al.³³

The probabilities $\mathcal{P}(t) = \text{Tr}(\Lambda W)$ are the most fundamental quantities in quantum physics. They describe the probability to register or measure the observable Λ in the state W and represent the directly measured experimental quantities; $\mathcal{P}(t)$ is measured by the number of counts of a registration apparatus (detector) and its derivative, $\dot{\mathcal{P}} = \frac{d}{dt}\mathcal{P}(t)$ is observed as the normalized counting rate of the detector. We want to apply these probabilities to the case where Λ is a projection operator (or more generally a positive definite operator) on a subspace of non-interacting decay products registered by a detector and W is a quasistable state $W^D(t)$. In this case $\mathcal{P}(t)$ is the decay probability and $\dot{\mathcal{P}}(t)$ is the decay rate for the decaying state W^D into the decay products Λ registered by the detector. Thus Λ is experimentally defined by the detector and $W^D(0)$ by the preparation apparatus or preparation process of the quasistable state (e.g. a resonance produced in a scattering experiment or a metastable product of some ancient creation process which is present at some initial time $t = 0$).

Traditionally, W is an ensemble defined by a macroscopic preparation apparatus (single microsystems are not defined) and Λ and W are mathematically represented by operators in the Hilbert space. However, as we mentioned in Section 3, the decay probabilities within the HS formulation have severe problems, justifying a fresh approach. Our approach is to extend the basic formula $\mathcal{P}(t) = \text{Tr}(\Lambda W)$ to apply to the decaying state W^D mathematically represented by the Gamow vector of Section 7.2, $W^D = |\psi^G\rangle\langle\psi^G|$. We give the following interpretation to this basic formula applied to W^D : a single microphysical decaying system described by W^D has been produced by a macroscopic apparatus and a quantum scattering process, at a time $t = 0$. Each count of the detector in a decay experiment is the result of the decay of this single microsystem that has lived for a time t_a —the time that it took the decaying system to travel from the scattering center to the decay vertex. The integral

$$\int_{t_a - \frac{1}{2}\Delta t}^{t_a + \frac{1}{2}\Delta t} dt \dot{\mathcal{P}}(t) \approx \dot{\mathcal{P}}(t_a)\Delta t,$$

³³p. 1345 of [13].

is proportional to the number of microsystems that have decayed in the time interval Δt , i.e. $\dot{\mathcal{P}}(t)$ is the normalized counting rate of the detector, $\dot{N}(t)/N(\infty)$. The lifetime τ_D (mean life) is the average of the t_a 's:

$$\sum_a t_a \Delta t \dot{\mathcal{P}}(t_a) \approx \int dt t \dot{\mathcal{P}}(t) = \tau_D.$$

This is the way $\dot{\mathcal{P}}(t)$, τ_D , etc. are obtained (defined) experimentally. The quantity $\mathcal{P}(t)$ in the basic formula is thus experimentally the normalized number of counts $N(t)/N(\infty)$ (with N very large). Therefore the normalization condition of the W^D is such that

$$\mathcal{P}(\infty) = 1; \quad (73)$$

the probability of finding the state decayed is certainty. In addition $\mathcal{P}(t)$ is subject to an initial boundary condition

$$\mathcal{P}(0) = 0; \quad (74)$$

the probability of finding one of the many (N) decay products already at $t = 0$ is zero. The reason for this is that $t = 0$ is the time at which the decaying state has been prepared and one starts counting the decay products (one assumes there are no other incoming decay products in the detector area).

In the mathematical theory of quantum mechanics, the quantity $\mathcal{P}(t)$ is calculated from the operators Λ and W^D by the basic formula (39a) given by

$$\mathcal{P}(t) = \text{Tr}(\Lambda W^D(t)) = \text{Tr}(\Lambda(t) W^D) = \langle \psi^D(t) | \Lambda | \psi^D(t) \rangle, \text{ for } t \geq 0 \text{ only.} \quad (75)$$

The formula for the decay rate $\dot{\mathcal{P}}(t)$ should be obtained as the derivative of the right hand side of (75).

However, in the standard textbooks of quantum mechanics one does not calculate the transition probabilities (75) ³⁴. Instead one gives a Golden Rule [55] for the initial decay rate $\dot{\mathcal{P}}(t = 0)$, and one justifies this Golden Rule with some ingenious heuristic arguments that were originated by Dirac [53]. In place of an initial ($t=0$) decaying state $W^D = |\psi^D\rangle\langle\psi^D|$ which evolves by the exact Hamiltonian $\psi^D(t) = e^{-iHt}\psi^D$, one usually chooses an eigenstate of the interaction-free Hamiltonian, $\psi^D \rightarrow f^D$, where

$$H_0 f^D = E_D f^D \text{ with } H_0 = H - V. \quad (76)$$

³⁴One cannot calculate it in the HS formulation because of (25), [33].

In place of the projection operators for decay products, one uses the “improper” states, $\Lambda \rightarrow |E_b, b\rangle\langle E_b, b|$ which have dimension of (energy) $^{-1}$. The Golden Rule is then given as the decay rate energy-density

$$\dot{\mathcal{P}}_b^{E_b}_D(0) = \frac{2\pi}{\hbar} |\langle b, E_b | V | f^D \rangle|^2 \delta(E_D - E_b), \quad (77a)$$

and the initial decay rate is obtained by integration and/or summation over all final quantum numbers E_b and b , as:

$$\dot{\mathcal{P}}_D(0) = \frac{2\pi}{\hbar} \int dE_b \sum_b |\langle b, E_b | V | f^D \rangle|^2 \delta(E_b - E_D). \quad (77b)$$

Alternatively, one chooses for the initial state also an improper “energy eigenstate” of the free Hamiltonian, $H_0 |E_D, d\rangle = E_D |E_D, d\rangle$, or of the exact Hamiltonian, $H |E_D, d^+\rangle = E_D |E_D, d^+\rangle$. Then, for example, one can write the exact but highly singular Golden Rule as

$$\begin{aligned} \dot{\mathcal{P}}_b^{E_b E_D}_D(0) &= \frac{2\pi}{\hbar} |T_{bD}|^2 \delta(E_D - E_b), \text{ where} \\ T_{bD} &= \langle b, E_b | V | E_D, d^+ \rangle \approx \langle b, E_b | V | E_D, d \rangle. \end{aligned} \quad (77c)$$

The initial transition probability rate $\dot{\mathcal{P}}(0)$ is then obtained by integration/summation over the final quantum numbers E_b and b and averaging over the initial quantum numbers E_D and d .

The expression (77b) (and its variants) is one of the most important and most widely used formulas in quantum physics. It expresses the decay probability per unit time of the state $|f^D\rangle\langle f^D|$ at $t = 0$ (the time when the decaying state has been created and the registration of the decay products begins) into the non-interacting decay products described by the projection operator given by Λ :

$$\Lambda = \sum_{\substack{\text{all } b \\ b \neq b^D}} \int_0^\infty dE |E, b\rangle\langle E, b|, \text{ where } H_0 |E, b\rangle = E |E, b\rangle \quad (78)$$

The initial decay rate $\dot{\mathcal{P}}(t = 0) = \frac{d}{dt} \mathcal{P}(t)|_{t \rightarrow 0^+}$ of (77b) should be the time derivative of the probability (75), at least approximately, or it should be the time derivative of a probability $\mathcal{P}(t) = \text{Tr}(\Lambda(t) |f^D\rangle\langle f^D|)$. This suggests the following program: Find a decaying state $W^D(t) = |\psi^D(t)\rangle\langle\psi^D(t)|$ that evolves in time according to the exact Hamiltonian $H = H_0 + V \geq 0$,

$$W^D(t) = e^{-iHt} W^D(0) e^{iHt}, \quad (79)$$

calculate the decay probability $\mathcal{P}(t)$ using the basic formula (75), and then take the time derivative of the probability, $\dot{\mathcal{P}}(t)$. This decay rate at $t = 0$, $\dot{\mathcal{P}}(0)$, should somehow resemble Dirac's Golden Rule (77b), at least in some approximation $\psi^D \rightarrow f^D$.

From the results discussed in Sect. 4, in particular from the result that $\mathcal{P}(t) \equiv 0$ for every $\psi^D \in \mathcal{H}$ [33], it is clear that such a program cannot be implemented by a mathematical theory in the Hilbert space. The futility of such attempts was the actual reason for the introduction of the Gamow vectors [24]. We shall now describe how the Gamow vectors and irreversible quantum mechanics in the RHS lead to a decay rate formula that reduces to Dirac's Golden Rule in the Born approximation.

For the detector Λ , one takes a positive operator

$$\Lambda = \sum_{\substack{\text{all } b \\ b \neq b^D}} \int_0^\infty dE \lambda_b(E) |E, b\rangle \langle E, b|, \quad (80)$$

where $H_0|E, b\rangle = E|E, b\rangle$ and where $\lambda_b(E)$ is a smooth and rapidly decreasing function of E (and of all the other (continuous) quantum numbers in b) that describes the detector efficiency. The summation in b extends over those quantum numbers (and momentum directions) of the decay products which are registered by the detector but the summation does not include the quantum numbers b^D of the decaying state.

For the decaying state $W^D(t) = |\psi^D(t)\rangle \langle \psi^D(t)|$ one takes the pure Gamow state ψ^G of Sect. 7.2:

$$\psi^D(t) = \frac{1}{f} \psi^G(t) \in \Phi_+^\times, \quad (81)$$

where the “normalization” factor f is chosen such that $\mathcal{P}(\infty) = 1$ when Λ describes *all* decay products. This state has the time evolution (57a), which in operator form is:

$$W^G(t) = |\psi^G(t)\rangle \langle \psi^G(t)| = e^{-iH^\times t} |\psi^G\rangle \langle \psi^G| e^{iHt} = e^{-\Gamma t} |\psi^G\rangle \langle \psi^G|, \text{ for } t \geq 0 \text{ only.} \quad (82a)$$

This is inserted for $W^D(t)$ in (75) where also $t \geq 0$. The equality (82a) is understood as a functional equation in the space Φ_+ , as in (57a) and (57b), i.e. as

$$\langle \psi_1^- | W^G(t) | \psi_2^- \rangle = e^{-\Gamma t} \langle \psi_1^- | \psi^G \rangle \langle \psi^G | \psi_2^- \rangle \text{ for } \psi_1^-, \psi_2^- \in \Phi_+. \quad (82b)$$

This means (82a) is only valid in matrix elements with the vectors $\psi^- \in \Phi_+$, which represent the out-states that are ultimately registered by the detectors. This presents no limitations, since only those matrix elements are experimentally accessible³⁵.

³⁵The matrix element $\langle \phi^+ | W^G | \phi^+ \rangle$ does not make any sense mathematically or physically.

The decay probability (75), with (80) and (81), can now be derived in a lengthy calculation [24] using (82). This derivation makes use of the Lippmann-Schwinger equation in one of its standard (singular) forms:

$$|E, b^-\rangle = |E, b\rangle + \frac{1}{E - H - i\epsilon} V |E, b\rangle \quad (83)$$

where $H^\times |E, b^-\rangle = E |E, b^-\rangle$ and $H_0^\times |E, b\rangle = E |E, b\rangle$. The derivation also makes use of the relation

$$\langle^- E | f(H^\times) | \psi^G \rangle = f(z_R) \langle^- E | \psi^G \rangle, \quad (84)$$

which follows from (55) and can be proven as a functional equation over the $\langle^- \psi | E^- \rangle \in (\mathcal{S} \cap \mathcal{H}_-^2)_{\mathbb{R}^+}$. From (83) and (84) one obtains for sufficiently good interaction Hamiltonians V (such that $\langle E | V | \psi^G \rangle \in (\mathcal{S} \cap \mathcal{H}_-^2)^\times$):

$$\langle \psi^G | E, b^- \rangle = \langle \psi^G | E, b \rangle + \langle \psi^G | V | E, b \rangle \frac{1}{E - (E_R + i\Gamma/2) - i\epsilon}, \quad (85)$$

This is again understood as a functional equation over the $\langle^- \psi | E^- \rangle \in (\mathcal{S} \cap \mathcal{H}_-^2)_{\mathbb{R}^+}$. With these equations, and under the assumption that the mathematically singular expressions above can be rigorously justified³⁶, one calculates the following result:

$$\mathcal{P}(t) = 1 - e^{-\Gamma t} \int_0^\infty dE \sum_{b \neq b^D} \lambda_b(E) |\langle E, b | V | \psi^G \rangle|^2 \frac{1}{(E - E_R)^2 + (\Gamma/2)^2}; \quad t \geq 0. \quad (86)$$

This is the probability for the transition of the decaying state W^D into all mixtures of decay products with the property Λ . Usually one omits the detector efficiency $\lambda_b(E)$ in formulas like (86) above and (87)–(89) and (98) below. This means one gives these formulas for an ideal detector with $\lambda_b(E) = 1$ for all b which are quantum numbers of decay products and with $\lambda_b(E) = 0$ otherwise. The detector efficiency $\lambda_b(E)$ is then used in the analysis of

³⁶In contrast to the statements 1. and 2. (eq.25) in Sect. 4), whose proofs use only well defined mathematics of the HS, the statement (86) cannot be formulated as a mathematical theorem because its derivation requires such singular expressions as the Lippmann-Schwinger equation. While this equation is exact and well-accepted by the physics community, it has as yet not been given a mathematically rigorous foundation in either the HS or RHS. In this respect (86) and (87) are also different from the statements in the preceding sections, like the semigroup time evolutions (45) and (46) in $\Phi_\mp \subset \mathcal{H} \subset \Phi_\mp^\times$, the exponential law (57) and (82), the Nuclear Spectral Theorem (60) and the “complex spectral theorem” (65) and (71), which are well-founded in the mathematics of the RHS. On the other hand, (87) reproduces in a reasonable approximation the well-proven Golden Rule, providing heuristic support.

the experimental data to correct the observed events for detector efficiency. The reported counting rate has usually been corrected for this detection efficiency. The factor $\lambda_b(E)$ is then omitted in the theoretical formulas like (86), but this factor is always present and can be used in the mathematical calculations to make expressions like (85) less singular.

Taking the time derivative of (86) (setting $\lambda_b(E) = 1$), we obtain for the decay rate

$$\dot{\mathcal{P}}(t) = e^{-\Gamma t} 2\pi \int_0^\infty dE \sum_{b \neq b^D} |\langle E, b | V | \psi^G \rangle|^2 \frac{\Gamma/2\pi}{(E - E_R)^2 + (\Gamma/2)^2}; \quad t \geq 0. \quad (87)$$

Since $\lambda_b(E)$ has been omitted the decay rate in (87) is to be compared to the experimental counting rate corrected for detection efficiency. The formula (87), and also (86), we call the exact Golden Rule.

The initial decay rate is then obtained as

$$\dot{\mathcal{P}}(0) = 2\pi \int_0^\infty dE \sum_{b \neq b^D} |\langle E, b | V | \psi^G \rangle|^2 \frac{\Gamma/2\pi}{(E - E_R)^2 + (\Gamma/2)^2}. \quad (88)$$

Since the probability to find the decay product at time $t \leq 0$ needs to be zero, $\mathcal{P}(0) = 0$, we obtain from (86) for $t = 0$:

$$\frac{2\pi}{\Gamma} \int_0^\infty dE \sum_{b \neq b^D} |\langle E, b | V | \psi^G \rangle|^2 \frac{\Gamma/2\pi}{(E - E_R)^2 + (\Gamma/2)^2} = 1. \quad (89)$$

The sums in (86) ... (89) extend over all quantum numbers and all decay products. Comparing (89) with (88), we obtain

$$\dot{\mathcal{P}}(0) = \Gamma \quad (= \Gamma/\hbar), \quad (90)$$

and from the exponential time dependence in (86) or (87) we obtain the lifetime of the decaying state:

$$\tau_G = \frac{1}{\Gamma} \quad (= \hbar/\Gamma), \quad (91)$$

The result (90) means that the imaginary part of the complex energy in (55), which is also the imaginary part of the S -matrix pole position and the width of the Breit-Wigner energy distribution (56) is equal to the initial rate $\dot{\mathcal{P}}(0)$ of the decay probability (75).

The formulas (86) and the formulas (87) and (88) give the *total* decay probability and the *total* decay rate, respectively, because we took for Λ the projection operator on *all* decay

products and summed over *all* values of quantum numbers $b = \{b_1, b_2, \dots, \eta\}$ of *all* the decay products. The partial decay rates are obtained if we sum in (87) and (88) only over part of the quantum numbers $b = \{b_1, b_2, \dots, \eta\}$, e.g. over all $\hat{b} = \{b_1, b_2, \dots\}$ but not over the quantum number η that characterizes the decay channels:

$$\dot{\mathcal{P}}_\eta(t) = 2\pi e^{-\Gamma t} \int_0^\infty dE \sum_{b_1, b_2, \dots} |\langle E, b_1, b_2, \dots, \eta | V | \psi^G \rangle|^2 \frac{\Gamma/2\pi}{(E - E_R)^2 + (\Gamma/2)^2}. \quad (92)$$

The value $\dot{\mathcal{P}}_\eta$ is the partial decay rate into decay products with the quantum number η or into the decay channel characterized by η . The total decay rate ((87) or (88)) is then written as the sum over partial decay rates, with a corresponding relation for the “partial widths” $\Gamma_\eta \equiv \hbar \dot{\mathcal{P}}_\eta(0)$:

$$\dot{\mathcal{P}}(t) = \sum_{\text{all } \eta} \dot{\mathcal{P}}_\eta(t), \quad \Gamma = \sum_{\text{all } \eta} \Gamma_\eta. \quad (93)$$

In analogy to (93), with (92) one can write for the decay probability (86) (setting $\lambda_b(E) = 1$)

$$\begin{aligned} \mathcal{P}(t) = 1 - \sum_\eta \frac{2\pi}{\Gamma} e^{-\Gamma t} \int_0^\infty dE \sum_{b_1, \dots, \neq b^D} |\langle E, b_1, \dots, \eta | V | \psi^G \rangle|^2 \\ \times \frac{\Gamma/2\pi}{(E - E_R)^2 + (\Gamma/2)^2}, \quad t \geq 0. \end{aligned} \quad (94)$$

The derivation of the above formulas (86), (87), (88) and (92) did not make use of any approximations. We therefore call these formulas exact Golden Rules. Equation (86) is the exponential decay law with the directionality of time [56]

$$\mathcal{P}(t) = 1 - e^{-\Gamma t}, \quad t \geq 0. \quad (95)$$

From the quantum mechanical arrow of time (Section 4), via the semigroup (46) and the evolution of the Gamow states (57a), we derived the time-asymmetry for the decay process: *If we start at $t = 0$ from a “decaying state” $|\psi^G\rangle\langle\psi^G|$, then the probability of finding the decay products Λ increases exponentially with time from $\mathcal{P}(0) = 0$ to $\mathcal{P}(t \rightarrow \infty) = 1$.*

In addition to these results concerning principles, we can also obtain a rule for the calculation of the initial decay rate or width in quantum theory. The exact Golden Rules are not of much practical use in calculations since these formula contain the unknown ψ^G and the natural line width under the integral. The state vector ψ^G is unknown and the values (E_R, Γ) are unknown until the eigenvalue equation

$$H^\times \psi^G = \left(E_R - i\frac{\Gamma}{2} \right) \psi^G; \quad \psi^G \in \Phi_+^\times \quad (96)$$

has been solved. Therefore, to obtain a calculational tool from the above relations (87), (88) and (92), one uses the Born approximation on the right hand side: one expands the exact ψ^G in a perturbation series (for the interaction Hamiltonian V) in terms of the non-interacting “decaying state vector” f^D defined by (76), since the solutions of (76) are usually known.

The Born approximation is given by

$$\langle b, E|V|\psi^D\rangle \approx \langle b, E|V|f^D\rangle \quad (97a)$$

$$\frac{\Gamma}{2E_R} \rightarrow 0 \quad (97b)$$

$$E_R \approx E_D \quad (97c)$$

The Breit-Wigner energy distribution (natural line width) has the property:

$$\lim_{\frac{\Gamma}{2E_R} \rightarrow 0} \frac{\Gamma/2\pi}{(E - E_R)^2 + (\Gamma/2)^2} = \delta(E - E_R). \quad (97d)$$

Using (97), one obtains from (88) the initial decay rate in this Born approximation:

$$\dot{\mathcal{P}}(0) = \frac{2\pi}{\hbar} \int dE \sum_b |\langle b, E|V|f^D\rangle|^2 \delta(E - E_D), \quad (98)$$

and similar expressions for the other formulas. This is the standard Golden Rule (77b) again, only here it has been *derived* from the transition probability (75) and from the time evolution of the Gamow vectors (82) as the Born approximation of an exact Golden Rule.

The Gamow vector ψ^G provides the link that was missing from the HS formulation to connect the all-important empirical rules for the rates to the fundamental theoretical relations for the probabilities.

8 Summary

Dirac’s bra and ket formalism can be given a mathematical justification by the rigged Hilbert space (RHS). The RHS does not only contain Dirac kets (“scattering states”) but also Gamow vectors, which are eigenkets $|E_R - i\Gamma/2\rangle$ of self-adjoint Hamiltonians H with complex eigenvalue $(E_R - i\Gamma/2)$ representing decaying states or resonances with Breit-Wigner energy distribution of energy E_R and width Γ . The value Γ is also the initial rate of the decay probability $\mathcal{P}(t)$, $\Gamma = \hbar \frac{d\mathcal{P}}{dt}|_{t=0}$, for which an exact Golden Rule could be derived. In the Born approximation, this exact Golden Rule becomes Dirac’s Golden Rule. These features are central requirements for a theory of resonances and decay, yet the standard quantum theory in

Hilbert space (HS) can not produce them since the transition probabilities from a decaying state $W^D(t)$ into the observed decay products Λ , $\mathcal{P}(t) = \text{Tr}(\Lambda W^D(t))$, can be proven to be identically zero if they are zero at any time interval, e.g. before the decaying state has been prepared. Therefore, resonances in the HS formulation of quantum mechanics cannot be described by a state vector $\psi^D \in \mathcal{H}$ and had to be denied the status of autonomous microphysical entities in HS. This is contrary to the phenomenological observation that stability and the value of the lifetime are not criteria for elementarity and that quasistable states and stable states should be described on the same footing—as it is done in S -matrix theory.

The semigroup time evolution of the Gamow states sprung from the new mathematical language of the RHS and led to their exact exponential decay with lifetime $\tau = \hbar/\Gamma$

$$e^{-iH^\times t}|E_R - i\Gamma/2\rangle = e^{-iE_R t}e^{-(\frac{\Gamma}{2})t}|E_R - i\Gamma/2\rangle, \text{ for } t \geq 0 \text{ only.}$$

The semigroup $e^{-iH^\times t}$, $t \geq 0$, expresses intrinsic irreversibility on the microphysical level. This irreversibility is not restricted to the time evolution of Gamow states. It can be formulated in terms of time-asymmetric boundary conditions for states and observables and is related to a general arrow of time which can be expressed by the truism: A state needs to be prepared first before an observable can be registered in it. This can also be expressed as a condition of time ordering $t_a > t_0$ in the probabilities $\mathcal{P}(t) = \text{Tr}(\Lambda(t_a)W(t_0))$, where $t = t_0$ is the time at which state W has been prepared and the registration of the observable Λ can begin (or the initial time of the universe $W(t_0) = \rho_i$ if the quantum mechanics of measured systems is extended to the quantum theory of cosmology). This intrinsic irreversibility seems to have a different origin than the standard extrinsic irreversibility of open quantum systems which results from the effect of an external reservoir or measurement apparatus. The latter has also a semigroup evolution which however is generated by a reservoir-dependent Liouvillian and not by the Hamiltonian of the quantum system. This intrinsic irreversibility also does not follow from the collapse axiom of a pure or less mixed state into a more mixed state. However, the change of state due to measurement scattering of a microsystem on a macrosystem possesses, like every scattering process, an arrow of time which has the same origin as the semigroup.

Since our time-asymmetry comes from the boundary conditions and the algebras of observables and the dynamical laws are time-symmetric, microphysical irreversibility is compatible with a time reversal transformation (or CPT) if one chooses for T a representation that doubles the space of states and observables. Such representations of T exist, but the interpretation of a time-reversal doubled world is not clear.

The Gamow kets describing Breit-Wigner resonances have been derived from first order poles of the analytically continued S -matrix. This can be generalized to quasi-stationary systems associated with higher order poles of the S -matrix. An S -matrix pole of r -th order

at $z_R = E_R - i\Gamma/2$ leads to r generalized eigenvectors of order $k = 0, 1, \dots, r - 1$. The Gamow vector of order k , $|z_R\rangle^{(k)}$, is the k -th derivative of the ordinary Gamow vector and, except for a normalization factor, also a Jordan vector of degree $(k + 1)$ with generalized eigenvalue $z_R = E_R - i\Gamma/2$.

The Gamow vectors appear in the complex basis vector expansion, which is, for state vectors, an alternative to the Dirac basis vector expansion and contains Gamow kets instead of Dirac kets as basis vectors. The complex basis vector expansion is particularly useful for problems that involve resonances and decay. For a system with a finite number of resonance states (such as the neutral Kaon system), it leads to the “effective” theories with complex Hamiltonian matrices (e.g. Lee-Oehme-Yang theory of the K_L - K_S system) by truncating the complex basis vector expansion to the subspace spanned by the Gamow vectors, the resonance subspace. In addition to the Gamow kets the complex basis vector expansion of a prepared state vector also contains other terms called background. This background term varies with the preparation process of the state and its properties are not connected with the decaying Gamow state (or states, if more than one resonance is involved). In particular, the time evolution of this preparation-dependent background term is non-exponential.

All that has been achieved by the RHS is a mathematical theory that allows to separate out the exponentially decaying (and also the exponentially growing which have not been discussed here) states and to derive their Golden Rule from the fundamental probabilities of quantum mechanics. In the process of doing this, the time-asymmetry of quantum physics became apparent.

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